Harmonizable Stable Processes With Rational Spectral Densities

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1 Introduction

Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a strongly harmonizable symmetric α stable process, $1 < \alpha \leq 2$, SH(SαS)P. Then $X(t)$ is the Fourier transform of a SαS random measure with independent increments Φ,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\Phi(d\lambda). \tag{1.1}$$

Then $f(\lambda) = \frac{\|\Phi(d\lambda)\|_\alpha^\alpha}{d\lambda}$, where $\|\cdot\|_\alpha$ is the Schilder's norm, defines the spectral density of the process [?]. In this work we assume that $f(\cdot)$ is a rational function. More precisely

$$f(\cdot) = \left|\frac{Q(\cdot)}{P(\cdot)}\right|^2$$

where $Q(\cdot)$ and $P(\cdot)$ are polynomials of $i\lambda$, with no common factors, that the degree$(Q(\cdot)) <$ degree$(P(\cdot))$.

The closed linear span of the $X(t),\ t \in \mathbf{R}$, under the $\|\cdot\|_\alpha$, denoted by $(\mathcal{A}, \|\cdot\|_\alpha)$, forms the time domain of the process which is a Banach space of jointly SαS random variables. By using the covariations $[X(t), X(S)]]_\alpha = f^\vee(t-s)$, the elements of the process can be equipped with an inner product, resulting the Hilbert space $(S, \langle\cdot,\cdot\rangle_S)$ generated by the processes ([?]), see Theorem 1.1 given below. Let $S^T =$ span S-closure$\{x(t) : |t| \leq T\}$. Then

$$S^{0+} = \cap_{T>0} S^T$$

is the S-Germ field of the process. Our aim is to show that the S-Germ field is finite dimensional and an analog of Hida's Theorem (1960) is also satisfied in S for the process $X(t)$.

Theorem 1.1. Let $X = \{X(t),\ t \in \mathbf{R}\}$ be a purely non deterministic SH(SαS)P given by (1.1), then there is a Hilbert space of jointly symmetric stable random variables denoted by $(\mathcal{S}, <\cdot,\cdot>_{\mathcal{S}})$, for which

(i) $\mathcal{S} \subset \mathcal{A}$, as a point inclusion,

(ii) $\|Y\|_\alpha \leq C\|Y\|_{\mathcal{S}}$, for every $Y \in \mathcal{S}$, where C is a constant independent of Y.

(iii) $Y \in \mathcal{S}$, $Y = \int gdM$ where $g \in \mathrm{L}^2$ and $M(A) = \int_A \frac{1}{h^*} d\Phi$, , and h is an outer function of class H^2 for which $f = |h|^2$.

(iv) For $Y_1 = \sum_{l=1}^n d_l X(t_l)$ and $Y_2 = \sum_{j=1}^m b_j X(s_j)$,

$$\begin{aligned}\langle Y_1, Y_2 \rangle &= \sum_{l,j} d_l b_j^* [X(t_l), X(s_j)]_\alpha \\ &= 2\pi \sum_{l,j} d_l b_j^* f^\vee(s_j - t_l),\end{aligned}$$

where $f^\vee(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-itu}du$ and * stands for the complex conjugate, also $\hat{f}(u) = \int_{-\infty}^{\infty} f(t)e^{itu}dt$ and $< \cdot, \cdot >_\alpha$ stands for the covariation.

Corollary 1.1 It follows from (iii) that $g_n(\cdot) \to g$ in L^2 if and only if $Y_n = \int g_n dM \to Y = \int gdM$ in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, and only if $Y_n \to Y$ in $\mathcal{A}$ Let us also record the following facts in L^2 theory concerning rational h

Lemma 1.1

Let $\mathcal{L}^T = \mathrm{spanL}^2 - \mathrm{clousure}\{e^{i\lambda t}h : |t| \leq T\}$ and $\mathcal{L}^{0+} = \cap_{T>0}\mathcal{L}^T$. Then $\mathcal{L}^{0+}$ is the class of polynomials of degree less than $d - n_0 - n_1$, Also $\partial^k h^\vee : 0 \leq k \leq d - n_0 - n_1 - 1$ is a basis for $\mathcal{L}^{0+}$, where $h = \frac{p_0 p_1}{p_2}$ of real polynomials of $i\lambda$ with no common factors of degrees n_0, n_1, n_2 respectively, that the roots of p_0 lie on the line and the roots of p_1 and p_2 in the lower half plane and $n_0 + n_1 < n_2 = d$.

Lemma 1.2 Let $D = (2\pi)^{-1/2}p(i\partial)$ be a real differentiable operator with degree d of constant coefficients, where $p(\lambda) = \sum_{n=0}^d c_n(-i\lambda)^n$. Also let $e_n : 0 \leq n < d$ be a basis for the solutions of $D[k] = 0$ that satisfies

$$\partial^k e_n(0+) = \begin{cases} 1 & \text{if } n = k < d \\ 0 & \text{if } n \neq k < d. \end{cases}$$

Then the solution of $D[k] = g$ with initial data $\partial^n k(0+) : 0 \leq n < d$ is

$$f(t) = \sum_{n=0}^{d-1} \partial^n f(0+)e_n(t) + \frac{1}{c_d}\int_0^t e_{d-1}(t-s)g(s)ds,$$

Lemma 1.3 Suppose D and p are as in Lemma 1.2 and $h = \frac{1}{p}$. Then $D[h^\vee] = 0$ for $t > 0$, and $\partial^n h^\vee(0+) = 0 : 0 \leq n \leq d-2$, and $\partial^{d-1}h^\vee(0+) = 1/c_d$

Lemma 1.4 Suppose h is as in Lemma 1.3, then $h^\vee = e_{d-1}/c_d$ where $e_n : 0 \leq n < d$ are as in Lemma 1.2.

Proof:

Apply Lemmas 1.2 and 1.3 and note that $g \equiv 0$. To furnish the ingrediants we recall the following reslts (NS)

Theorem 1.2. Let $X = \{X(t), t \in \mathbf{R}\}$ be a purely non deterministic SH(SαS)P, then

$$X(t) = \int_{-\infty}^{t} h^{\vee}(t-s)dZ(s), \quad t \in \mathbf{R}. \tag{1.2}$$

in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$, and consequently in $(\mathcal{A}, \|\cdot\|_{\alpha})$, where for bounded Borel sets $A \subset \mathbf{R}$, $Z(A) = \int \hat{I}_A(\lambda)dM(\lambda)$. Furthermore $\langle Z(A), Z(B)\rangle_{\mathcal{S}} = 0$, $A \cap B = \phi$, and $\mathcal{S}_t(X) = \mathcal{S}_t(\Delta Z)$, $t \in \mathbf{R}$, where $\mathcal{S}_t(X) = \overline{\text{sp}}\{X(s),\ s \le t\}$ in $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ and $\mathcal{S}_t(\Delta Z) = \overline{\text{sp}}\{Z(A) :\ A \subset (-\infty, t]$ and A is bounded $\}$.

2 The Germ field $\mathcal{S}^{0+}$

In this section it is to be proved that the Germ field $\mathcal{S}^{0+}$ is the finite dimensional subspace generated by SαS random variables

$$\partial^k X(0) : 0 \le k \le d - n_0 - n_1 - 1.$$

Let us first prove that these random variables are well defined.

Lemma 2.1 Let h be as in Lemma 1.1, then $\partial^k X(0) : 0 \le k \le d - n_0 - n_1 - 1$ are well defined SαS random variables that generate the Germ field S^{0+}.
Proof.
Since $\lambda^k h(\lambda) : 0 \le k \le d - n_0 - n_1 - 1$ are in Ł^2, by using the inequality

$$\left| \int \left[e^{i\lambda t} d\lambda - \sum_{k=0}^{n} \frac{(it)^k}{k!}\lambda^k \right] h(\lambda)d\lambda \right| \le \int \min\left\{ \frac{|t\lambda|^{n+1}}{(n+1)!}, \frac{2|t\lambda|^n}{n} \right\} h(\lambda)d\lambda,$$

for any $t \in R$, the Dominated Convergence Theorem implies that

$$\partial^k h^{\vee}(t) = i^k \int \lambda^k e^{i\lambda t} h(\lambda)d\lambda, \quad t > 0$$

and

$$\partial^k h^{\vee}(0+) = i^k \int \lambda^k h(\lambda)d\lambda$$

in Ł^2. Therefore by Colorrary 1.1

$$\partial^k X(t) = i^k \int \lambda^k e^{i\lambda t} h(\lambda)dM, \quad 0 \le k \le d - n_0 - n_1 - 1, \tag{2.1}$$

and

$$\partial^k X(0+) = i^k \int \lambda^k h(\lambda)dM, \quad 0 \le k \le d - n_0 - n_1 - 1$$

To characterize $\mathcal{S}^{0+}$, note that by Theorem 1.1(iii) $\mathcal{S}^{0+}$ is isomorphic to $\mathcal{L}^{0+}$, then apply Lemma 1.1. The proof is complete.

The following theorem is an analoge of the fundamental result of Hida(1960)

Theorem 2.1

Suppose $X(t)$ is a SH(SαS)P for which $f(\lambda) = |p(\lambda)|^{-2}$ where $p(\lambda)$ is as in Lemma 1.2. Then $D[X] = \overset{\bullet}{Z}$ where is the differential operator given in Lemma 1.2 and $\overset{\bullet}{Z}$ is the (so called) derivative of FN SαS process given in Theorem 1.2.

Proof.

It follows from Theorem 1.2 that

$$\begin{aligned} X(t) &= \int_{-\infty}^{t} h^{\vee}(t-s)dZ(s) \\ &= \int_{-\infty}^{0} h^{\vee}(t-s)dZ(s) + \int_{0}^{t} h^{\vee}(t-s)dZ(s) \\ &= \int_{-\infty}^{0} h^{\vee}(t-s) \overset{\bullet}{Z}(s)ds + \int_{0}^{t} h^{\vee}(t-s) \overset{\bullet}{Z}(s)d(s) \end{aligned}$$

where $\overset{\bullet}{Z}(t)$ is an $\langle\cdot,\cdot\rangle_{\mathcal{S}}$-orthogonal S$\alpha$S process in $\mathcal{S}$ and is considered as the (generalized) derivative of the FN Process $Z(t)$. Now apply Lemma 1.4 to observe that

$$X(t) = \int_{-\infty}^{0} h^{\vee}(t-s) \overset{\bullet}{Z}(s)ds + \frac{1}{c_d}\int_{0}^{t} e_{d-1}(t-s) \overset{\bullet}{Z}(s)d(s),$$

where $e_n : 0 \leq n < d$ are as in Lemma 1.2. Since by Lemma 1.3 $D[h^{\vee}] = 0$, it follows from Lemma 1.2 that

$$\int_{-\infty}^{0} h^{\vee}(t-s) \overset{\bullet}{Z}(s)ds = \sum_{n=0}^{d-1} \partial^n X(\cdot)e_n(t).$$

The existence of $\partial^n X(0)$ was established in the proof of Theorem 2.1. Therefore

$$X(t) = \sum_{n=0}^{d-1} \partial^n X(\cdot)e_n(t) + \frac{1}{c_d}\int_{0}^{t} e_{d-1}(t-s) \overset{\bullet}{Z}(s)d(s),$$

giving the result.

Colorrary 2.1

Suppose $X(t)$ is a SH(SαS)P for which $f(\lambda) = \left|\frac{p_0(\lambda)p_1(\lambda)}{p_2(\lambda)}\right|^2$ where $p_0(\lambda)$, $p_1(\lambda)$, $p_2(\lambda)$ are as in Lemma 1.1. Then

$$X(t) = \sum_{n=0}^{n_0+n_1} b_n \partial^n Y(t) \tag{2.2}$$

where $\sum_{n=0}^{n_0+n_1} b_n(i\lambda)^n = p_0(\lambda)p_1(\lambda)$ and $Y(t)$ is the SH(SαS)P with spectral density $|p_2(\lambda)|^{-2}$.

Proof.
Observe that

$$\begin{aligned} X(t) &= \int e^{it\lambda} \frac{p_0(\lambda)p_1(\lambda)}{p_2(\lambda)} M(d\lambda) \\ &= \sum_{n=0}^{n_0+n_1} b_n \int (i\lambda)^n e^{it\lambda} \frac{1}{p_2(\lambda)} M(d\lambda). \end{aligned}$$

Apply (2.1) with $n_0 = n_1 = 0$ and $d = n_2$ to arrive at (2.2).

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