Harmonizable Stable Processes With Rational Spectral Densities

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1 Introduction

Let $X = \{X(t), \ t \in \mathbb{R}\}$ be a strongly harmonizable symmetric $\alpha$ stable process, $1 < \alpha \leq 2$, SH(SoS)P. Then $X(t)$ is the Fourier transform of a SoS random measure with independent increments $\Phi$,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} \Phi(d\lambda).$$  \hfill (1.1)

Then $f(\lambda) = \|\Phi(d\lambda)\|_{\alpha}$, where $\| \cdot \|_{\alpha}$ is the Schilder’s norm, defines the spectral density of the process \cite{[1]}. In this work we assume that $f(\cdot)$ is a rational function. More precisely

$$f(\cdot) = \left| \frac{Q(\cdot)}{P(\cdot)} \right|^2$$

where $Q(\cdot)$ and $P(\cdot)$ are polynomials of $i\lambda$, with no common factors, that the degree($Q(\cdot)$) < degree($P(\cdot)$).

The closed linear span of the $X(t)$, $t \in \mathbb{R}$, under the $\| \cdot \|_{\alpha}$, denoted by $(\mathcal{A}, \| \cdot \|_{\alpha})$, forms the time domain of the process which is a Banach space of jointly SoS random variables. By using the covariances $[X(t), X(S)]_{\alpha} = f^\vee(t-s)$, the elements of the process can be equipped with an inner product, resulting the Hilbert space $(\mathcal{S}, \langle \cdot, \cdot \rangle_s)$ generated by the processes $([\cdot])$, see Theorem 1.1 given below. Let $S^T = \text{span } S$-closure\{x(t) : |t| \leq T\}. Then

$$S^{0+} = \cap_{T>0} S^T$$

is the $S$-Germ field of the process. Our aim is to show that the $S$-Germ field is finite dimensional and an analog of Hida’s Theorem (1960) is also satisfied in $S$ for the process $X(t)$.

**Theorem 1.1.** Let $X = \{X(t), \ t \in \mathbb{R}\}$ be a purely non deterministic SH(SoS)P given by (1.1), then there is a Hilbert space of jointly symmetric stable random variables denoted by $(\mathcal{S}, < \cdot, \cdot >_s)$, for which
(i) $\mathcal{S} \subset \mathcal{A}$, as a point inclusion,
(ii) $\|Y\|_\alpha \leq C\|Y\|_\mathcal{S}$, for every $Y \in \mathcal{S}$, where $C$ is a constant independent
of $Y$.
(iii) $Y \in \mathcal{S}$, $Y = \int g dM$ where $g \in L^2$ and $M(A) = \int_A \frac{1}{h} d\Phi$, and $h$ is an
outer function of class $H^2$ for which $f = |h|^2$.
(iv) For $Y_1 = \sum_{i=1}^n \delta_i X(t_i)$ and $Y_2 = \sum_{j=1}^m b_j X(s_j)$,
\[\langle Y_1, Y_2 \rangle = \sum_{i,j} d_i b_j^* [X(t_i), X(s_j)]_\alpha\]
\[= 2\pi \sum_{i,j} d_i b_j^* f^\alpha(s_j - t_i),\]
where $f^\alpha(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-i \alpha u} du$ and $^\ast$ stands for the complex conjugate, also
\[\hat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{i \alpha u} dt\] and $< \cdot, \cdot >_\alpha$ stands for the covariation.

**Corollary 1.1** It follows from (iii) that $g_n(\cdot) \to g$ in $L^2$ if and only if $Y_n = \int g_n dM \to Y = \int g dM$ in $(\mathcal{S}, < \cdot, \cdot >_\mathcal{S})$, and only if $Y_n \to Y$ in $\mathcal{A}$ Let us also record the following facts in $L^2$ theory concerning rational $h$

**Lemma 1.1**

Let $\mathcal{L}^T = \text{span} L^2 \to \text{closure}\{e^{i\lambda} h : |\lambda| \leq T\}$ and $\mathcal{L}^{0+} = \cap_{T>0} \mathcal{L}^T$. Then $\mathcal{L}^{0+}$ is the class of polynomials of degree less than $d - n_0 - n_1$. Also $\partial^k h^\alpha : 0 \leq k \leq d - n_0 - n_1 - 1$ is a basis for $\mathcal{L}^{0+}$, where $h = \frac{e_p}{p_2}$ of real polynomials of $i\lambda$ with no common factors of degrees $n_0, n_1, n_2$ respectively, that the roots of $p_0$ lie on the line and the roots of $p_1$ and $p_2$ in the lower half plane and $n_0 + n_1 < n_2 = d$.

**Lemma 1.2** Let $D = (2\pi)^{-1/2} p(i\partial)$ be a real differentiable operator with degree $d$ of constant coefficients, where $p(\lambda) = \sum_{n=0}^d c_n (-i\lambda)^n$. Also let $e_n : 0 \leq n < d$ be a basis for the solutions of $D[k] = 0$ that satisfies

\[\partial^k e_n (0+) = \begin{cases} 1 & \text{if } n = k < d \\ 0 & \text{if } n \neq k < d. \end{cases}\]

Then the solution of $D[k] = g$ with initial data $\partial^0 k(0+) : 0 \leq n < d$ is

\[f(t) = \sum_{n=0}^{d-1} \partial^n f(0+) e_n(t) + \frac{1}{c_d} \int_0^t e_{d-1}(t-s) g(s) ds,\]

**Lemma 1.3** Suppose $D$ and $p$ are as in Lemma 1.2 and $h = \frac{1}{p}$. Then $D[h^\alpha] = 0$ for $t > 0$, and $\partial^n h^\alpha(0+) = 0 : 0 \leq n \leq d - 2$, and $\partial^{d-1} h^\alpha(0+) = 1/c_d$

**Lemma 1.4** Suppose $h$ is as in Lemma 1.3, then $h^\alpha = e_{d-1}/c_d$ where $e_n : 0 \leq n < d$ are as in Lemma 1.2.

**Proof:**

Apply Lemmas 1.2 and 1.3 and note that $g \equiv 0$. To furnish the ingredients
we recall the following results (NS)

**Theorem 1.2.** Let $X = \{X(t), t \in \mathbb{R}\}$ be a purely non deterministic SH(SoS)P, then

$$X(t) = \int_{-\infty}^{t} h^\vee(t-s)dZ(s), \ t \in \mathbb{R}. \quad (1.2)$$

in $(\mathcal{S}, \langle \cdot, \cdot \rangle_{\mathcal{S}})$, and consequently in $(\mathcal{A}, \|\cdot\|_{a})$, where for bounded Borel sets $A \subset \mathbb{R}$, $Z(A) = \int \hat{I}_A(\lambda)dM(\lambda)$. Furthermore $\langle Z(A), Z(B) \rangle_{\mathcal{S}} = 0$, $A \cap B = \phi$, and $\mathcal{S}(X) = \mathcal{S}(\Delta Z)$, $t \in \mathbb{R}$, where $\mathcal{S}(X) = \mathcal{S}_p\{X(s), s \leq t\}$ in $(\mathcal{S}, \langle \cdot, \cdot \rangle_{\mathcal{S}})$ and $\mathcal{S}(\Delta Z) = \mathcal{S}_p\{Z(A) : A \subset (-\infty, t] \text{ and } A \text{ is bounded}\}$.

## 2 The Germ field $\mathcal{S}^{0+}$

In this section it is to be proved that the Germ field $\mathcal{S}^{0+}$ is the finite dimensional subspace generated by SoS random variables

$$\partial^k X(0) : 0 \leq k \leq d - n_0 - n_1 - 1.$$  

Let us first prove that these random variables are well defined.

**Lemma 2.1** Let $h$ be as in Lemma 1.1, then $\partial^k X(0) : 0 \leq k \leq d - n_0 - n_1 - 1$ are well defined SoS random variables that generate the Germ field $\mathcal{S}^{0+}$.

**Proof.**

Since $\lambda^k h(\lambda) : 0 \leq k \leq d - n_0 - n_1 - 1$ are in $L^2$, by using the inequality

$$\left| \int e^{it\lambda}d\lambda - \sum_{k=0}^{n} \frac{(it)^k}{k!} \lambda^k \right| h(\lambda)d\lambda \leq \int \min \left\{ \frac{|t\lambda|^{n+1}}{(n+1)!}, \frac{2|t\lambda|^n}{n} \right\} h(\lambda)d\lambda,$$

for any $t \in R$, the Dominated Convergence Theorem implies that

$$\partial^k h^\vee(t) = i^k \int \lambda^k e^{i\lambda^t}h(\lambda)d\lambda, \ t > 0$$

and

$$\partial^k h^\vee(0+) = i^k \int \lambda^k h(\lambda)d\lambda$$

in $L^2$. Therefore by Corollary 1.1

$$\partial^k X(t) = i^k \int \lambda^k e^{i\lambda t}h(\lambda)dM, \ 0 \leq k \leq d - n_0 - n_1 - 1, \quad (2.1)$$

and

$$\partial^k X(0+) = i^k \int \lambda^k h(\lambda)dM, \ 0 \leq k \leq d - n_0 - n_1 - 1.$$
To characterize $S^{0+}$, note that by Theorem 1.1(iii) $S^{0+}$ is isomorphic to $L^{0+}$, then apply Lemma 1.1. The proof is complete.

The following theorem is an analogue of the fundamental result of Hida (1960)

**Theorem 2.1**

Suppose $X(t)$ is a SH(SαS)P for which $f(\lambda) = |p(\lambda)|^2$ where $p(\lambda)$ is as in Lemma 1.2. Then $D[X] = \dot{Z}$ where is the differential operator given in Lemma 1.2 and $\dot{Z}$ is the (so called) derivative of FN SαS process given in Theorem 1.2.

**Proof.**

It follows from Theorem 1.2 that

\[
X(t) = \int_{-\infty}^{t} h^\gamma(t - s) dZ(s) \\
= \int_{-\infty}^{0} h^\gamma(t - s) dZ(s) + \int_{0}^{t} h^\gamma(t - s) dZ(s) \\
= \int_{-\infty}^{0} h^\gamma(t - s) \dot{Z}(s) ds + \int_{0}^{t} h^\gamma(t - s) \dot{Z}(s) ds
\]

where $\dot{Z}(t)$ is an $(\cdot, \cdot)_S$-orthogonal SαS process in $S$ and is considered as the (generalized) derivative of the FN Process $Z(t)$. Now apply Lemma 1.4 to observe that

\[
X(t) = \int_{-\infty}^{0} h^\gamma(t - s) \dot{Z}(s) ds + \frac{1}{c_d} \int_{0}^{t} e_{d-1}(t - s) \dot{Z}(s) ds,
\]

where $e_n : 0 \leq n < d$ are as in Lemma 1.2. Since by Lemma 1.3 $D[h^\gamma] = 0$, it follows from Lemma 1.2 that

\[
\int_{-\infty}^{0} h^\gamma(t - s) \dot{Z}(s) ds = \sum_{n=0}^{d-1} \partial^n X(\cdot)e_n(t).
\]

The existence of $\partial^n X(0)$ was established in the proof of Theorem 2.1. Therefore

\[
X(t) = \sum_{n=0}^{d-1} \partial^n X(\cdot)e_n(t) + \frac{1}{c_d} \int_{0}^{t} e_{d-1}(t - s) \dot{Z}(s) ds,
\]

giving the result.

**Corollary 2.1**

Suppose $X(t)$ is a SH(SαS)P for which $f(\lambda) = \left| \frac{p_0(\lambda)p_1(\lambda)}{p_2(\lambda)} \right|^2$ where $p_0(\lambda)$, $p_1(\lambda)$, $p_2(\lambda)$ are as in Lemma 1.1. Then

\[
X(t) = \sum_{n=0}^{n_0+n_1} b_n \partial^n Y(t)
\]  

(2.2)
where $\sum_{n=0}^{n_0+n_1} b_n(i\lambda)^n = p_0(\lambda)p_1(\lambda)$ and $Y(t)$ is the SH(SaS)P with spectral density $|p_2(\lambda)|^{-2}$.

**Proof.**

Observe that

$$X(t) = \int e^{it\lambda} \frac{p_0(\lambda)p_1(\lambda)}{p_2(\lambda)} M(d\lambda)$$

$$= \sum_{n=0}^{n_0+n_1} b_n \int (i\lambda)^n e^{it\lambda} \frac{1}{p_2(\lambda)} M(d\lambda).$$

Apply (2.1) with $n_0 = n_1 = 0$ and $d = n_2$ to arrive at (2.2).

**References**


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