Two-Stage Confidence Intervals for the Variance of a Normal Distribution

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1. Abstract

In this paper, we design a concrete fixed-width confidence interval for the variance σ^2 of a normal distribution $N(\mu,\sigma^2)$ by two-stage procedure and provide the optimal sample size of the first stage by numerical computations. Moreover, for fixed samples, we establish a direct proof for the non-existence of fixed-width confidence interval for σ^2 by Bayesian method.

Keywords: Confidence interval, Sequential Confidence Intervals, Two-stage Confidence Intervals, Bayesian Solution, Minimax Solution.

MR(1991) Subject Classification: 62F25.

2. Introduction

Let $N(\mu,\sigma^2)$ denote a normal distribution, where μ is mean and σ^2 is variance and both are unknown throughout this paper. We want to find a confidence interval of prescribed width L and prescribed coverage probability $1-\alpha$ for the mean μ. Dantzig (1940) prove that: suppose $X_1,\ldots,X_n$ are a sequence of independent samples from $N(\mu,\sigma^2)$, let $X=(X_1,\ldots,X_n)$, if we stipulate that $\left[m(X)-\frac{L(X)}{2},m(X)+\frac{L(X)}{2}\right]$ is a confidence interval for m with confidence level $g>0$, then $\sup_{x} L(x)=\infty$. This result shows that no fixed sample procedure can meet our demands. Accordingly, sequential procedure has to be used. Stein [1] (1945) proposed a two-stage procedure and Zeng Jianjun [2] studied the optimal sample size of the first stage. Simons and Starr carried out researches into the asymptotically efficient properties of fixed-width sequential confidence intervals, further, Chow and Robbins [3] generalized the problem by considering an arbitrary distribution instead of normal distribution.

Let (Ω ,F) be a measurable space, on which P is a class of probability measures and

X_1, X_2, Λ are a sequence of independent identically distributed variables taking values in R^k for any $P \in$ P. Let F_P denote the distribution and distribution function of X_1 on (R^k,B^k). Suppose $h(P)$ is a finite, real valued function on P and e is a set of confidence intervals for $h(P)$ basing on$\{X_i\}$. We say that $h(P)$ can be estimated precisely in e and write $h(p)\in e$ if for all $L>0$ and $a>0$, there exists a confidence interval in e such that the width of which is less than L and the confidence level of which is $1-a$. Let e_m and e_∞ denote the confidence intervals set of m -stage procedure and sequential procedure respectively. Bahadur and Savage [4] proved that $h(P)\notin e_\infty$ when $h(P)$ is the mean of F_P and { $F_P, P\in$ P } is the set of all one-dimension

distributions with finite mean. Singh [5] obtained the necessary conditions for $h(P)\in e_1$ (fixed samples procedure) and $h(P)\in e_\infty$. Chen Xiru [6][7] supplied the sufficient conditions for $h(P)\in e_1, h(P)\in e_\infty$ and $h(P)\in e_2$, by which we know the variance $s^2\in e_2$.

So far, however, the concrete confidence interval for $h(P)$ (for instance, the variance s^2) has not been seen as many in actual problems. In this paper, we construct a fixed-width confidence interval for the variance of a normal distribution by two-stage procedure step by step and give the optimal sample size of the first stage by numerical computations. Further, for fixed samples, we establish a direct proof of the non-existence of fixed-width confidence interval for the variance by Bayesian method. Theoretically speaking, though we can get the proof by testing the necessary conditions stated in [5], it is not so easy.

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3. Two-Stage ProcedureSubtitle

In this section, using two-stage procedure, we design a confidence interval whose width is less than L and confidence level is $1-a$ for the variance s^2 of a normal distribution $N(m,s^2)$ through four steps.

Step 0. **Let** $\{X_n, n \geq 1\}$ **be a sequence of real valued, independent identically distributed samples**

from $N(m, s^2)$. **Put** $\sqrt{1-a} = 1 - b$, **take** $c^2_{n,1,\frac{b}{2}}$ **and** $c^2_{n,2,\frac{b}{2}}$ **such that** $P\left(c^2(n) \leq c^2_{n,1,\frac{b}{2}}\right) = \frac{b}{2}$,

$P\left(c^2(n) \geq c^2_{n,2,\frac{b}{2}}\right) = \frac{b}{2}$ **(we use notation** $P(F(\cdot) \in A)$, **where** $F(\cdot)$ **is a distribution function and**

A **is a event set, to denote** $P(x \in A)$, **where** x **is a random variable abiding by** $F(\cdot)$ **).**

Step 1. **Take** n_1 **samples:** $X_1, X_2, \Lambda, X_{n_1} \sim N(m, s^2)$, **we have**

$$P_{m,s^2}\left(s^2_{1n_1} \leq s^2 \leq s^2_{2n_1}\right) = 1 - b = \sqrt{1-a} \geq 1 - a, \qquad (2.1)$$

where

$$\overline{X_{n_1}} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i, \quad Sn_1^{\,2} = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(X_i - \overline{X_{n_1}}\right)^2, \quad s^2_{1n_1} = \frac{(n_1-1)Sn_1^{\,2}}{c_{n_1-1,2,\frac{b}{2}}}, \quad s^2_{2n_1} = \frac{(n_1-1)Sn_1^{\,2}}{c_{n_1-1,1,\frac{b}{2}}}.$$

Step 2. **If** $s^2_{2n_1} - s^2_{1n_1} \leq L$, **we stop sampling and have got confidence interval** $[s^2_{1n_1}, s^2_{2n_1}]$ **which**

satisfies the given conditions. Otherwise, we continue to take samples X_{n_1+1}, X_{n_1+2}, Λ , $X_{n_1+n_2}$

for the second stage, where n_2 **is determined by following parts. Write** $\overline{X_{n_2}} = \frac{1}{n_2}\sum_{i=1}^{n_2} X_{n_1+i}$,

$Sn_2^{\,2} = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(X_{n_1+i} - \overline{X_{n_2}}\right)^2$. **Supposing**

$$P_{m,s^2}\left(s^2 - Sn_2^{\,2} > C_{n_2,b}\right) = \frac{b}{2} \qquad (2.2)$$

$$\Leftrightarrow P_{m,s^2}\left(\frac{s^2}{n_2 - 1}\left(\frac{1}{s^2}\sum_{i=1}^{n_2}\left(X_{n_1+i} - \overline{X_{n_2}}\right)^2 - (n_2 - 1)\right) < -C_{n_2,b}\right) = \frac{b}{2}$$

$$\Leftrightarrow P_{m,s^2}\left(c^2(n_2 - 1) < (n_2 - 1) - \frac{n_2 - 1}{s^2}C_{n_2,b}\right) = \frac{b}{2},$$

so we get

$$C_{n_2,b} = s^2\left(1 - \frac{1}{n_2 - 1}c^2_{n_2-1,1,\frac{b}{2}}\right). \qquad (2.3)$$

Similarly, supposing

$$P_{m,s^2}\left(s^2 - Sn_2^{\,2} < C'_{n_2,b}\right) = \frac{b}{2}, \qquad (2.4)$$

we obtain

$$C'_{n_2,\beta} = \sigma^2\left(1-\frac{1}{n_2-1}\chi^2_{n_2-1,2,\frac{\beta}{2}}\right). \tag{2.5}$$

Combining (2.2), (2.4), we have

$$P_{\mu,\sigma^2}\left(Sn_2^{\ 2}+C'_{n_2,\beta} \le \sigma^2 \le Sn_2^{\ 2}+C_{n_2,\beta}\right)=1-\beta=\sqrt{1-\alpha} \tag{2.6}$$

and the width of the confidence interval $C_{n_2,\beta}-C'_{n_2,\beta}$. Let

$$\tilde{C}'_{n_2,\beta} = \sigma^2_{2n_1}(1-\frac{1}{n_2-1}\chi^2_{n_2-1,2,\frac{\beta}{2}}) \text{ and } \tilde{C}_{n_2,\beta} = \sigma^2_{2n_1}(1-\frac{1}{n_2-1}\chi^2_{n_2-1,1,\frac{\beta}{2}}).$$

Now, choose the smallest n_2 such that

$$\tilde{C}_{n_2,\beta}-\tilde{C}'_{n_2,\beta} \le L. \tag{2.7}$$

Step 3. Calculate the interval

$$[\sigma^2_{1*},\sigma^2_{2*}]\overset{\Delta}{=}[\sigma^2_{1n_1},\sigma^2_{2n_1}]\cap[Sn_2^{\ 2}+\tilde{C}'_{n_2,\beta}, Sn_2^{\ 2}+\tilde{C}_{n_2,\beta}]. \tag{2.8}$$

Theorem 1. The interval $[\sigma^2_{1*},\sigma^2_{2*}]$ is the confidence interval whose width is less than L and confidence level is $1-\alpha$.

Proof. By (2.7), (2.8), we have $\sigma^2_{2*}-\sigma^2_{1*} \le \tilde{C}_{n_2,\beta}-\tilde{C}'_{n_2,\beta} \le L$. Next, notice $C'_{n_2,\beta}<0$, $C_{n_2,\beta}>0$, by which we deduce

$$P_{\mu,\sigma^2}\left(\sigma^2_{1*}\le\sigma^2\le\sigma^2_{2*}\right)$$

$$=P_{\mu,\sigma^2}\left(\sigma^2_{1n_1}\le\sigma^2\le\sigma^2_{2n_1}, Sn_2^{\ 2}+\tilde{C}'_{n_2,\beta}\le\sigma^2\le Sn_2^{\ 2}+\tilde{C}_{n_2,\beta}\right)$$

$$\ge P_{\mu,\sigma^2}\left(\sigma^2_{1n_1}\le\sigma^2\le\sigma^2_{2n_1}, Sn_2^{\ 2}+C'_{n_2,\beta}\le\sigma^2\le Sn_2^{\ 2}+C_{n_2,\beta}\right)$$

$$=P_{\mu,\sigma^2}\left(\sigma^2_{1n_1}\le\sigma^2\le\sigma^2_{2n_1}\right)\times P_{\mu,\sigma^2}\left(Sn_2^{\ 2}+C'_{n_2,\beta}\le\sigma^2\le Sn_2^{\ 2}+C_{n_2,\beta}\right)$$

$$=\sqrt{1-\alpha}\sqrt{1-\alpha}=1-\alpha.$$

In this way, we see that the proof will be established as long as (2.7) hold. However, (2.7) holds surely when n_2 is sufficient large. Using the Central Limit Theorem,

$$\frac{\chi^2_{n_2-1}-(n_2-1)}{\sqrt{2(n_2-1)}}\xrightarrow{d}N(0,1) \text{ as } n\to\infty.$$

Hence, when n_2 is large enough, we have

$$P\left(\frac{\chi^2_{n_2-1}-(n_2-1)}{\sqrt{2(n_2-1)}}\ge u_{\frac{\beta}{2}}\right)=\frac{\beta}{2} \text{ and } P\left(\frac{\chi^2_{n_2-1}-(n_2-1)}{\sqrt{2(n_2-1)}}\le -u_{\frac{\beta}{2}}\right)=\frac{\beta}{2},$$

where $u_{\frac{b}{2}}$ denotes $P\left(N(0,1) \geq u_{\frac{b}{2}}\right) = \frac{b}{2}$. Thus

$$c^2_{n_2-1,1,\frac{b}{2}} = \left(-\sqrt{2(n_2-1)}u_{\frac{b}{2}} + (n_2-1)\right)(1+o(1)),$$

$$c^2_{n_2-1,2,\frac{b}{2}} = \left(\sqrt{2(n_2-1)}u_{\frac{b}{2}} + (n_2-1)\right)(1+o(1)),$$

which implies

$$\tilde{C}_{n_2,b} - \tilde{C}'_{n_2,b} = \frac{2\sqrt{2}}{\sqrt{n_2-1}} u_{\frac{b}{2}} s^2_{2n_1}(1+o(1))$$

$$\to 0 \text{ as } n \to \infty.$$

The proof then follows.

4. The Optimal Sample Size of the First Stage

In the course of above two-stage procedure, n_1 can be an arbitrary integer, how to choose it? In this section, we will work out approximately the optimal one by numerical computations.

Continue to use above notations, moreover, write $\tilde{L} = \frac{L}{s^2}$ and let N denote the total sample size of the two stages and $I(A)$ denotes the indicator function, where A is an event set:

$$A = \left(s^2_{2n_1} - s^2_{1n_1} \leq L\right) = \left(c^2(n_1-1) \leq \tilde{L}\frac{c^2_{n_1-1,1,\frac{b}{2}}\, c^2_{n_1-1,2,\frac{b}{2}}}{c^2_{n_1-1,2,\frac{b}{2}} - c^2_{n_1-1,1,\frac{b}{2}}}\right),$$

then $$N = n_1 I(A) + (n_1+n_2) I(A^c). \tag{3.1}$$

Since n_2 is a random variable and its distribution is relevant to m, s^2, a, L and n_1, the mean total size of samples can be written as

$$EN = E[n_1 I(A) + (n_1+n_2)I(A^c)] = n_1 + E[n_2 I(A^c)]$$

$$\stackrel{\Delta}{=} y(m, s^2, a, L, n_1). \tag{3.2}$$

Supposing

$$y(m, s^2, a, L, n_1^*) = \inf_{2 \leq n_1} y(m, s^2, L, a, n_1), \tag{3.3}$$

we then call n_1^* the optimal sample size of the first stage. Define function

$$T = j_b(n) = \frac{1}{n}\left(c^2_{n,2,\frac{b}{2}} - c^2_{n,1,\frac{b}{2}}\right), \quad (n \in N). \tag{3.4}$$

So (2.7) is equivalent to $s^2_{2n_1} j_b (n_2 - 1) \le L$. Using computer plotting, we know T is a stringently decreasing function of n for any $b > 0$, by which we can choose n_2 such that $s^2_{2n_1} j_b (n_2 - 1) = L$ without losing the essence. Therefore, for fixed $b > 0$,

$$n_2 = j^{-1}\left(\frac{L}{s^2_{2n_1}}\right) + 1 = j^{-1}\left(\tilde{L}\frac{c^2_{n_1-1,1,\frac{b}{2}}}{c^2(n_1-1)}\right) + 1. \qquad (3.5)$$

In order to calculate EN, the inverse function j^{-1} has to be worked out. However, it's very difficult to obtain its analytical solution, so we use numerical method.

In the sequel of this section, by *Mathematica* Programming, we present our results only for two cases: a=0.05 and a=0.01.

(i) If a=0.05, $b = 1 - \sqrt{1-a}$ =0.0253. By calculating (3.4), we get a sequence of points $\{(n, j_{b=0.0253}(n)), n \ge 1\}$. Inverting the coordinates, we now plot points $\{(j_{b=0.0253}(n), n), n \ge 1\}$ in Descartes coordinate planes as are shown in Figure 1. Fitting these points by the Least Squares method, we acquire the curve function

$$n = 39.7611 T^{-2}, \qquad (3.6)$$

that is also shown in Figure 1.

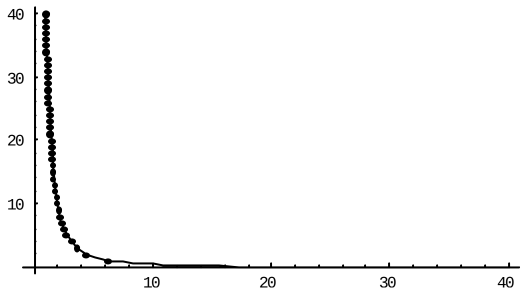

Figure 1

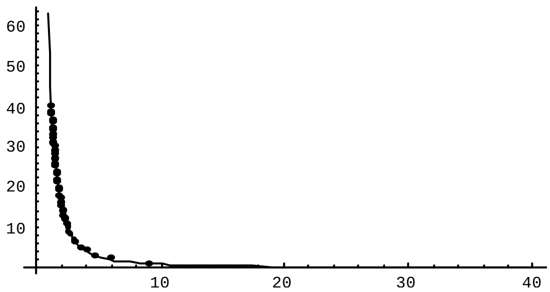

Figure 2

Beyond our expectation, the curve fits these points so well that it's reasonable to replace j^{-1} with (3.6). Substituting (3.5) and (3.6) into (3.2), we have

$$EN = n_1 + \frac{39.7611(n_1^2 - 1)}{\left(\tilde{L} c^2_{n_1-1,1,\frac{b}{2}}\right)^2}\left[1 - \int_0^{k(\tilde{L},n_1,b)} f(x; n_1+3)dx\right] + \left[1 - \int_0^{k(\tilde{L},n_1,b)} f(x; n_1-1)dx\right], \quad (3.7)$$

where $k(\tilde{L}, n_1, b) = \tilde{L}\dfrac{c^2_{n_1-1,1,\frac{b}{2}} c^2_{n_1-1,2,\frac{b}{2}}}{c^2_{n_1-1,2,\frac{b}{2}} - c^2_{n_1,1,\frac{b}{2}}}$, $f(x; n)$ denotes the density function of $c^2(n)$.

Totally similar to the case $a = 0.05$, when $a = 0.01$, then $b = 0.005013$, we get

$$n = 63.212 T^{-2}, \qquad (3.8)$$

that is shown in Figure 2 and

$$EN = n_1 + \frac{63.212(n_1^2-1)}{\left(\tilde{L}c^2_{n_1-1,1,\frac{b}{2}}\right)^2}\left[1-\int_0^{k(\tilde{L},n_1,b)} f(x;n_1+3)dx\right]+\left[1-\int_0^{k(\tilde{L},n_1,b)} f(x;n_1-1)dx\right]. \quad (3.9)$$

From (3.7) and (3.9), we see that EN is irrelevant to m, individual s^2 and individual L, only depending on $\tilde{L}\left(=\frac{L}{s^2}\right)$, a (through b) and n_1. Accordingly, (3.2) can be modified as

$$EN = y(\tilde{L}, a, n_1). \quad (3.10)$$

(ii) Now, by (3.7) and (3.9), we plot the cubic figures of EN, from which, we know, for fixed $\tilde{L}$, there exists an optimal n_1^* to minimize EN. For $a=0.05$, $a=0.01$ and $\tilde{L}$ in the range $[0.1,500]$, we calculate n_1^* which are presented as follows:

$\tilde{L}$	n_1^*	$y(\tilde{L}, a=0.05, n_1^*)$	$\tilde{L}$	n_1^*	$y(\tilde{L}, a=0.01, n_1^*)$
0.1	4399	4468.84	0.1	6822	6906.88
0.2	1180	1218.22	0.2	1813	1859.88
0.3	558	585.562	0.3	853	885.19
0.4	334	354.49	0.4	506	531.511
0.5	227	243.552	0.5	342	362.344
0.6	167	181.072	0.6	251	267.596
0.7	130	142.13	0.7	194	208.768
0.8	105	116.044	0.8	157	169.502
0.9	88	97.292	0.9	131	141.839
1.0	76	83.9997	1.0	111	121.513
1.1	66	73.6319	1.1	97	106.063
1.2	59	65.5257	1.2	86	94.0104
1.3	53	59.0278	1.3	77	84.3841
1.4	48	53.7241	1.4	70	76.5567
1.5	44	49.3281	1.5	64	70.075
1.6	41	45.638	1.6	59	64.6373
1.7	38	42.4841	1.7	54	60.0313
1.8	35	39.7958	1.8	51	56.0484
1.9	33	37.4358	1.9	48	52.6146
2.0	32	35.3816	2.0	45	49.6035
3.0	21	23.5001	3.0	30	32.4743
4.0	17	18.3419	4.0	23	25.0075.
5.0	14	15.361619	5.0	19	20.8692
6.0	12	13.5176	6.0	17	18.221

7.0	11	12.1699	7.0	15	16.372
8.0	10	11.2269	8.0	14	14.9963
9.0	10	10.4917	9.0	13	13.9376
10.0	9	9.81765	10.0	12	13.1227
15.0	8	8.1648	15.0	10	10.5419
20.0	7	7.14542	20.0	9	9.274
30.0	6	6.09742	30.0	8	8.0712
40.0	5	5.46628	40.0	7	7.1172
60.0	5	5.01893	60.0	6	6.17733
100.044.19304100.055.48449					
500.0	3	3.09376	500.0	4	4.00361

iii) We now fit the data above to acquire the empirical formula n_1^* of $\tilde{L}$, by which we can easily get the optimal sample size of the first stage for given $\tilde{L}$ in actual problems. Figure 3 and Figure 4 show the fit curves for $a = 0.05$ and $a = 0.01$ respectively whose function are also given below.

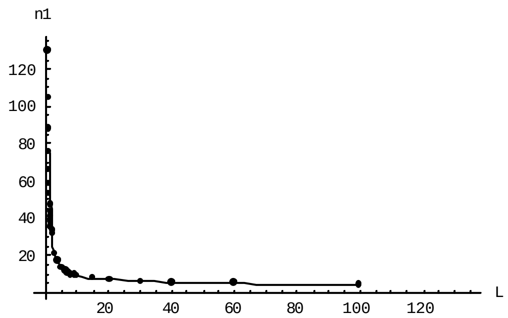

Figure 3

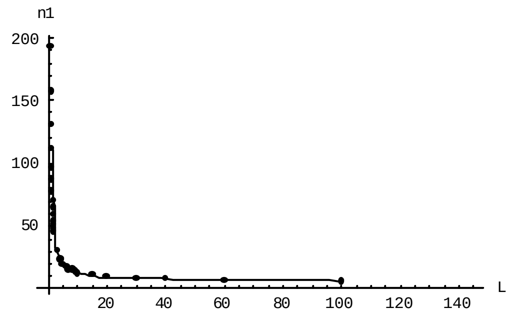

Figure 4

$$n_1^*\left(\tilde{L}\right) = 2.58701 - \frac{0.153208}{\tilde{L}^3} + \frac{43.5065}{\tilde{L}^2} + \frac{15.2609}{\tilde{L}} + \frac{14.658}{\sqrt{\tilde{L}}}, \ (a = 0.05\ ) \qquad (3.11)$$

$$n_1^*\left(\tilde{L}\right) = 3.14196 - \frac{0.181758}{\tilde{L}^3} + \frac{67.2134}{\tilde{L}^2} + \frac{21.5773}{\tilde{L}} + \frac{20.0742}{\sqrt{\tilde{L}}}. \ (a = 0.01) \qquad (3.12)$$

Notice, the programs we created can be applied for any $a > 0$.

iv) The Proof of the Non-existence of Fixed-width Confidence Interval Basing on Fixed Samples for s^2

Considering the confidence interval for the variance s^2 basing on fixed samples, another version of Dantzig Theorem holds, that is $s^2 \notin e_1$.

Theorem 2. Suppose X_1, Λ, X_n are a sequence of independent samples from $N\left(0, s^2\right)$, let X =(X_1, Λ, X_n), if we stipulate that $\left[a(X), b(X)\right]$ is a confidence interval for s^2 whose confidence level is 1- a , then $\sup_{x} (b(x)-a(x))=\infty$.

The following propositions are required for the proof of Theorem 2.

Proposition 1. Suppose $X_1,\Lambda\ ,X_n$ are a sequence of independent samples from $N\left(0,s^{2}\right)$, parameter s^{2} >0 is unknown, let $q=\frac{1}{s^{2}}$, choose the prior distribution of q abiding by $G(q;r,s)$ = $\frac{r^{s}}{\Gamma(s)}q^{s-1}e^{-rq}$, q >0, r>0, s>0. Let the loss function be $Lf\left(\frac{1}{q},[a,b]\right)=q(b-a)+mI_{[a,b]^c}(\frac{1}{q})$, where m is a positive constant and $[a,b]$ is a confidence interval for $\frac{1}{q}(=s^{2})$ and $I_{[a,b]^c}(\frac{1}{q})=\begin{cases}0 & a\le\frac{1}{q}\le b\\ 1 & otherwise\end{cases}$, then the Bayesian solution of $[a,b]$ has the form $[Sn^{*2}a_0,Sn^{*2}b_0]$, where $Sn^{*2}=\sum_{i=1}^{n}X_i^{2}$, $b_0>a_0>0$ are determined by

$$\frac{(b_0-a_0)(n+2s)}{1+2\frac{r}{Sn^{*2}}}+m\left(1-P\left(\frac{1}{b_0}\le G(\cdot;\frac{1}{2}+\frac{r}{Sn^{*2}},\frac{n}{2}+s)\le\frac{1}{a_0}\right)\right) =$$

$$\inf_{b>a>0}\left\{\frac{(b-a)(n+2s)}{1+2\frac{r}{Sn^{*2}}}+m\left(1-P\left(\frac{1}{b}\le G(\cdot;\frac{1}{2}+\frac{r}{Sn^{*2}},\frac{n}{2}+s)\le\frac{1}{a}\right)\right)\right\}. \qquad (4.1)$$

Proof. On account of $\frac{Sn^{*2}}{s^{2}}\sim c^{2}(n)$, put $Sn^{*2}=Y$, then the joint distribution

$$(Y,q)\sim\frac{y^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{r^{s}}{\Gamma(s)}e^{-(\frac{y}{2}+r)q}q^{\frac{n}{2}+s-1}.$$

Hence, the posterior distribution of q is that $q\ \ |\ Y\sim dH(q\,|\,y)=G(q;r+\frac{y}{2},\frac{n}{2}+s)$. Under the loss function Lf , we calculate the posterior risk as follows:

$$C([a,b],y)=\int_0^{\infty}Lf\,(\frac{1}{q},[a,b])dH(q\,|\,y)$$

$$=(b-a)\frac{n+2s}{y+2r}+m(1-P(\frac{y}{b}\le qy\le\frac{y}{a})) .$$

Notice $q\,|\,Y\sim G(\cdot;r+\frac{y}{2},\frac{n}{2}+s)$, which implies $qY\ |\ Y\ \sim G(\cdot;\frac{1}{2}+\frac{r}{y},\frac{n}{2}+s)$. Let the Bayesian solution of $[a,b]$ be $[\,ya_0,yb_0\,]$, then a_0,b_0 have the values such that condition (4.1) holds, which establishes the proof.

Proposition 2. Let m increases from $\dfrac{n\Gamma(\frac{n}{2})2^{\frac{n}{2}}e^{\frac{n}{2}+1}}{(n+2)^{\frac{n}{2}+1}}$ to ∞, suppose $b_m > a_m > 0$, satisfy condition (4.2), then a_m decreases from $\dfrac{1}{n+1}$ to 0, b_m increases from $\dfrac{1}{n+1}$ to ∞.

$$n(b_m - a_m) + m(1 - P(\frac{1}{b_m} \le c^2(n) \le \frac{1}{a_m}))$$

$$= \inf_{b>a>0} \{ (b-a) + m(1 - P(\frac{1}{b} \le c^2(n) \le \frac{1}{a})) \}. \qquad (4.2)$$

Proof. Write $g(a,b) = n(b-a) + m(1 - P(\frac{1}{b} \le c^2(n) \le \frac{1}{a}))$, then

$$\frac{\partial g}{\partial a} = -n + m\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}e^{-\frac{1}{2a}}\left(\frac{1}{a}\right)^{\frac{n}{2}+1} \text{ and } \frac{\partial g}{\partial b} = n - m\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}e^{-\frac{1}{2b}}\left(\frac{1}{b}\right)^{\frac{n}{2}+1}.$$

First, studying the function $f(x) = e^{-\frac{1}{2x}}\left(\frac{1}{x}\right)^{\frac{n}{2}+1}$, $(x>0)$, we have

$$f'(x)\begin{cases} >0 & n < \dfrac{2}{n+2} \\ =0 & n = \dfrac{2}{n+2} \\ <0 & n > \dfrac{2}{n+2} \end{cases}.$$

Next, $\lim_{x\to 0^+} f(x) = 0, \lim_{x\to+\infty} f(x) = 0$. Hence $f_{\max} = f\left(\dfrac{1}{n+2}\right) = e^{-\frac{n+2}{2}}(n+2)^{\frac{n}{2}+1}$.

Now, let $\dfrac{\partial g}{\partial a} = 0$, $\dfrac{\partial g}{\partial b} = 0$, we then have that when $m > \dfrac{n\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}e^{\frac{n+2}{2}}}{(n+2)^{\frac{n}{2}+1}}$, $a, b(b>a)$ has unique solution a^*, b^* respectively and $a^* < \dfrac{1}{n+2}$, $b^* > \dfrac{1}{n+2}$. Moreover, when m increases from $\dfrac{n\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}e^{\frac{n+2}{2}}}{(n+2)^{\frac{n}{2}+1}}$ to ∞, then a^* decreases from $\dfrac{1}{n+2}$ to 0, b^* increases from $\dfrac{1}{n+2}$ to ∞. Verifying a^* and b^* further, we know they are the minimum value point. This completes the proof.

By proposition 2, we can choose $m > 0$ and construct a confidence interval $[Sn^{*2}a_m, Sn^{*2}b_m]$ for s^2 such that its confidence coefficient is $1-a$, where a_m, b_m satisfies the condition (4.2). Fixes this m, under the loss function Lf (m has been chosen by virtue of above way), the risk function of $[Sn^{*2}a_m, Sn^{*2}b_m]$ is as follows:

$$R\left(s^2, [Sn^{*2}a_m, Sn^{*2}b_m]\right)$$

$$=n\left(b_m - a_m\right) + m\left(1 - \left(\frac{1}{b_m} \le c^2(n) \le \frac{1}{a_m}\right)\right) \stackrel{\Delta}{=} R.$$

Notice, the risk function is a constant (be irrelevant to s^2).

Proposition 3. Under the loss function Lf, $[Sn^{*2}a_m, Sn^{*2}b_m]$ above is the Minimax solution.

Proof. By proposition 1, we get the Bayesian risk of $\left[Sn^{*2}a_{r,s}, Sn^{*2}b_{r,s}\right]$

$$R_{r,s} = \int_0^{\infty} \left[\frac{(b_{r,s} - a_{r,s})(n+2s)}{1+\dfrac{2r}{y}} + m(1 - P\left(\frac{1}{b_{r,s}} \le G\left(\cdot, \frac{1}{2} + \frac{r}{y}, \frac{n}{2} + s\right) \le \frac{1}{a_{r,s}}\right) \right] dP_y,$$

where dP_y denotes the marginal probability distribution of $Y(\stackrel{\Delta}{=} Sn^{*2})$.

Let $r \to 0, s \to 0$ and choose a subsequences $\{b_{r_i,s_i} - a_{r_i,s_i}\}$ in $\{b_{r,s} - a_{r,s}\}$ such that

$$\lim_{i\to\infty}\left(b_{r_i,s_i} - a_{r_i,s_i}\right) = b_1 - a_1 \le \infty.$$

If $b_1 - a_1 = \infty$, the following result holds, so we suppose $b_1 - a_1 < \infty$ and then have

$$\lim_{i\to\infty} R_{r_i,s_i} = \lim_{i\to\infty} \int_0^{\infty} \left[\frac{\left(b_{r_i,s_i} - a_{r_i,s_i}\right)(n+2s_i)}{1+\dfrac{2r_i}{y}} + m\left(1 - P\left(\frac{1}{b_{r_i,s_i}} \le G\left(\cdot; \frac{1}{2} + \frac{r_i}{y}, \frac{n}{2} + s_i\right) \le \frac{1}{a_{r_i,s_i}}\right)\right) \right] dP_y$$

$$=n\left(b_1 - a_1\right) + m\left(1 - P\left(\frac{1}{b_1} \le c^2(n) \le \frac{1}{a_1}\right)\right)$$

$\ge R.$ (according to the definition of R.)

Via Theorem 2.3.6[1] (p_{154}), we get the proof.

From above discussions, we know that there exists $m > 0$ such that the confidence coefficient of $[Sn^{*2}a_m, Sn^{*2}b_m]$ is $1-a$ and $[Sn^{*2}a_m, Sn^{*2}b_m]$ is also the Minimax solution under the loss function Lf . Now suppose $[\bar{j},\underline{j}]$ is an arbitrary confidence interval for s^2 whose confidence level is $1-a$, we then have

$$\sup_{s^2>0}\left\{E_{s^2}\left[\frac{\bar{j}-\underline{j}}{s^2}\right]+mP_{s^2}\left(s^2\notin[\bar{j},\underline{j}]\right)\right\}$$

$$\geq\sup_{s^2>0}\left\{E_{s^2}\left[\frac{(b_m-a_m)Sn^{*2}}{s^2}\right]+mP_{s^2}\left(s^2\notin[Sn^{*2}a_m,Sn^{*2}b_m]\right)\right\}$$

$$=n(b_m-a_m)+ma.$$

Next, $[\bar{j},\underline{j}]$ has confidence level $1-a$, which implies $\sup_{s^2>0} mP_{s^2}\left(s^2\notin[\bar{j},\underline{j}]\right)\leq ma$. So that we obtain

$$\sup_{s^2>0}\frac{E_{s^2}[\bar{j}-\underline{j}]}{s^2 n(b_m-a_m)}\geq 1. \qquad (4.3)$$

Proof of Theorem 2. (By contradiction.) Suppose $\sup_x(b(x)-a(x))=d<\infty$, we then can choose one s_0^2 sufficient large such that $\frac{d}{s_0^2 n(b_m-a_m)}<1$. So when $s^2=s_0^2$, the contradiction arises against (4.3). This establishes the proof of Theorem 2.

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