The consumption-tracking problem of macroeconomic systems

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1. Intruduction. Input-output approach has been widely used as an analytical tool since it was developed by Leontief(1941) .The difficulty in the research on the Leontief dynamic input-output model is that, in general, this model is described as a singular difference equation or a singular differential equation , hence its solution contains the non-causal composition or the impulsive behaviour which is not suitable for using standard control techniques.

2. The optimal consumption-tracking problem. It is known that the supply available for external consumption and the external demand are not in the equilibrium, hence, the planner of government is in duty bound to try to minimize excess demands and to make a flexible control tactics A natural approach is to use the linear quadratic cost and to solve the LQ problem of the singular dynamic input-output model as follows, minimize

$$J_N=\sum_{k=0}^{N}(e'(k)Q(k)e(k)+u'(k)R(k)u(k)) \quad (1)$$

subject to

$Bx(k+1)=Bx(k)+u(k)$ ....(2) $y(k)=(I_n-A)x(k)-u(k)$ ....(3)

$e(k)=d(k)-y(k)$ ....(4) $Bx(0)=\alpha_0$ ....(5)

where, $x(k)\in R^n$ is a vector of production outputs, $y(k)\in R^n$ is a final consumption vector, $u(k)\in R^n$ is a control vector, $A\in R^{n\times n}$ is a production coefficient matrix, and $B\in R^{n\times n}$ is a capital coefficient matrix. The matrices A and B are constant and known, and the consumption vector y(k) is known . Usually, the matrix I_n-A is nonsingular, and the matrix B is singular. This is because an element bij of B represents the amount of capitalstock of output i that sector j needs to be able to produce one unit ofoutput. Since some sectors, agriculture is a typical example, do not produce capital goods, there will be rows with only zero elements. In such cases B will be singular. x(0) is the vector of production outputs in the initial year, Q(k) and R(k) are time-varying weighting matrices. We call this problem the optimal consumption-tracking problem.
Denote:

$$G=\begin{bmatrix}0 & 0\\ 0 & I_{n-p}\end{bmatrix},\ B_1=[I_p\ 0],\ B_2=[0\ I_{n-p}],\ D=I_n-A-G$$

$$B=\begin{bmatrix}B_{11} & B_{12}\\ 0 & 0\end{bmatrix},\ E=\begin{bmatrix}B^{-1}_{11} & -B_{11}^{-1}B_{12}\\ 0 & I_{n-p}\end{bmatrix} \quad (6)$$

where $B_1\in R^{p*n}$, $B_2\in R^{(n-p)*n}$ $B_{11}\in R^{p*p}$。

Make the full rank coordinate transformation: $Z(k)=E^{-1}X(k)=[Z_1(k)\ Z_2(k)]$, where $Z_1(k)\in R^p$, $Z_2(k)\in R^{n-p}$. And introduce the artificial feedback: $v(k)=u(k)-Gx(k)=u(k)-GEz(k)$

Rewrite the optimal consumption-tracking problem (1)-(5) as follows:
Minimize

$$J_{N,1}=\sum_{k=0}^{N}\xi'(k)W(k)\xi(k) \quad (7)$$

subject to ,

$\xi(k)=[Z_1'(k)\ d'(k)\ v'(k)]'$......(8) $Z_1(k+1)=Z_1(k)+B_1v(k)$ ......(9)

$0=Z_2(k)+B_2v(k)$ ...... (10) $y(k)=DEZ(k)-V(k)$......(11) $Z_1(0)=B_1\alpha_0$........(12)

where

$$W(k)=\begin{bmatrix}W_1(k) & W_2(k)\\ W_2'(k) & W_3(K)\end{bmatrix} \quad (13) \qquad W_1(k)=\begin{bmatrix}W_{11} & W_{12}\\ W_{12}' & Q(k)\end{bmatrix} \quad (14)$$

$W_{11}=B_1E'(D'Q(k)D+G'R(k)G)EB_1'$ (15) $W_{12}=-B_1E'D'Q(k)$ (16)

$W_2(k)=[W_{21}(k)\ W_{22}(k)]'$ (17) $W_{21}(k)=-[D'Q(k)(I_n+DEG)+G'R(k)(I_n-GEG)]'EB_1'$ (18)

$W_{22}(k)=(I_n+DEG)'Q(k)$ (19) $W_3(k)=(I_n+DEG)'Q(k)(I_n+DEG)+(I_n-GEG)'R(k)(I_n-GEG)$ (20) It is easy to see that: $\min J_N=\min J_{N,1}$. Notice that the positive definite property of the submatrix $w_3(k)$ plays an important role in minimizing the cost $J_{N,1}$. Investigating $W_3(k)$, further, we have:

Lemma l. Let $Q(k)>0$, $R(k)>0$, and suppose that the matrix I_n-A is nonsingular. Then the matrix $W_3(k)>0$.
Considering $W_3(k)>0$, we denote $v_1(k)=v(k)+w^{-1}_3(k)(w_{21}(k)z_1(k)+w_{22}(k)d(k))$ (21)

and rewrite the problem (7)-(12) as follows,minimize

$$J_{N.2}=\sum_{k=0}^{N}(y_1(k)-d_1(k))'W_4(k)(y_1(k)-d_1(k))+\sum_{k=0}^{N}v_1'(k)w_3(k)v_1(k) \quad (22)$$

subject to

$$Z_1(k+1)=A_1(k)Z_1(k)+B_1V_1(k)+F(k)d(k) \quad (23) \qquad y_1(k)=[I_p\ 0]'Z_1(k) \quad (24)$$

$$d_1(k)= -[0\ I_n]'d(k) \quad (25) \qquad Z_1(0)=B_{10}(0) \quad (26)$$

where, $[I_p\ 0] \in R^{p(n+p)}$, $[0\ I_n] \in R^{n(n+p)}$, and $W_4(k)=W_1(k)-W_2(k)W_3^{-1}(k)W'_2(k)$, $A_1(k)=I_p-B_1W_3^{-1}(k)W_{21}(k)$, $F(k)=-B_1W_3^{-1}(k)W_{22}(k)$.And we have,min $J_{N.1}$=min $J_{N.2}$.

Obviously, the problem (22)-(26) is an optimal trackingproblemof the regular state-space system . It can besolvedeasily.Furthermore, notice that in the cost (22), $y_1(k)$, k=0,1,... . , N, are only depending on $v_1(0),v_1(1),\ldots$,and $v_1(N-1)$, but not on $v_1(N)$,Hence,$v_1(k)$,k=0,1,....,N-1,and v1(N), can be optimized separately,andoptimizing$v_1(k)$,k=0,1,...,N-1,concludestosolvethefollowingstandardtrackingproblem, minimize:

$$J_{N.3}=(y_1(N)-d_1(N))'W_4(N)(y_1(N))-d_1(N))+\sum_{k=0}^{N-1}(y_1(k)-d_1(k))'W_4(k)(y_1(k)-d_1(k))+\sum_{k=0}^{N-1}v_1'(k)W_3(k)v_1(k) \quad (27)$$

subject to (23)-(26).

Denote $v_1^*(k)$ as the optimal control of the standard tracking problem above. Based on the classical results of linear optimal control theory, we have:

$$V_1^*(k)= -R_1^{-1}(k)B_1'(p(k+1)A_1(k)z_1(k)+p(k+1)F(k)d(k)+b(k+1)) \quad (28)$$

Where

$$R_1(k)=W_3(k)+B_1'p(k+1)B_1 ,\ p(N)=I_p[0\ W_4(N)I_p\ 0'],\ b(N)=I_p[0\ W_4(N)0\ I_n']d(N)$$

$$p(k)=A_1'(k)(p(k+1)-p(k+1)B_1R_1^{-1}(k)B_1'p(k+1))A_1(k)+I_p[0\ W_4(k)I_p\ 0']$$

$$b(k)=A_1'(k)(I_p-p(k+1)B_1R_1^{-1}(k)B_1')(p(k+1)F(k)d(k)+b(k+1)) +I_p[0\ W_4(k)0\ I_n']d(k)$$

Denote u*(k), y* (k), and x*(k) as the optimalcapitalstock, the final consumption, and the optimal productionoutputoftheoptimal consumption-trackingproblem(1)-(5),respectively. Obviously, u*(k), y*(k), and x*(k) can be computed. Then we have

$$u^*(k)=(G-L(k)B_1B)x^*(k)-M(k)d(k)-R_1^{-1}(k)B_1'b(k+1) \quad (29)$$

$$x^*(k)=\begin{bmatrix} L_{11}(k) & L_{12}(k) & L_{13}(k) \\ L_{21}(k) & L_{22}(k) & L_{23}(k) \end{bmatrix}\begin{bmatrix} z_1(k) \\ b(k+1) \\ d(k) \end{bmatrix} \quad (30)$$

$$y^*(k)=(D+L(k)B_1B)x^*(k)+M(k)d(k)+R_1^{-1}(k)B_1'b(k+1) \quad (31)$$

where, $L(k)=W^{-1}_3(k)W_{21}(k)+R^{-1}_1(k)B_1'p(k+1)A_1(k)$, $M(k)= W^{-1}_3(k) W_{22}(k)+ R^{-1}_1 (k) B_1'p(k+1)F(k)$ $L_{11}(k)=B^{-1}_{11}(I_p-B_{12}B_2L(k))$, $L_{12}(k)=-B^{-1}_{11}B_{12}B_2R_1^{-1}(k)B'_1$, $L_{13}(k)=-B^{-1}_{11}B_{12}B_2M(k)$, $L_{21}(k)=B_2L(k)$, $L_{22}(k)=B_2R_1^{-1}(k)B_1'$, $L_{23}(k)=B_2M(k)$,

$Z^*_1(k+1)=(I_p-B_1R_1^{-1}(k)B_1'p(k+1)(A_1(k)z^*_1(k)+F(k)d(k))-B_1R_1^{-1}(k)B'_1b(k+1)$, $z^*_1(0)=B_{10}$(0) WeSummarize the results above as the following theorem:

Theorem 1. Let Q(k)>0,R(k)>0, and suppose that the matrix I_n-A is nonsingular. Then the optimal consumption-trackingproblemofthe singular dynamic input-output model, i,e. the problem(1)-(5) , is solvable. The optimal capital stock, the available supply, and theoptimal production output can computed by using the equations (29)-(31). Andthe optimal capital stock can be synthesized as the linear feedback ofthe production output with a compensator related to the esternal demand.

Résumé : La article discuter essentiel rapport entre consommer de macroeconomique et general systéme LQ probléme. En ce qui concerne dynamique énergie et moderne singulier systéme résoudre bien la probléme