A Dynamic Martingales Approach For Defenses Against Ballistic Missiles

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1. Introduction and the Martingales Set-up

A considerable interest is growing among many countries to acquire the technology to produce ballistic missiles with nuclear, chemical, biological, or conventional warheads capable of reaching any part of the world. It becomes an essential military issue for other countries to protect themselves from such an attack. Hence, they built several defensive systems such as the PATRIOT by U.S.A. It is the objectives of this work to analyze the effectiveness of defenses against ballistic missiles and to construct useful bounds on the effectiveness of those defenses that are based on the notion of stochastic ordering.

Consider a random phenomenon that consists of a countable collection of experiments that are performed sequentially in time. Let $X_n(w)$ denote the outcome of experiment $n \in N^+ = (1, 2, \ldots)$. For convenience, we suppress w in $X_n(w)$. Set $X_0 \in N^+$. Let (Ω, F, P) denote the probability space and $F_n = \sigma(X_k, 0 \leq k \leq n)$ is the history of the process up to time n. Suppose that X_n is the sum of the X_{n-1} Bernoulli random variables with $Z(i,n) = 1$ if trial i results in a success at time n and $Z(i,n) = 0$ otherwise $(i = 1, \ldots, X_{n-1})$. These random variables are not necessarily independent nor identically distributed. It is understood that $X_n = 0$ whenever $X_{n-1} = 0 \ \forall \ n$. It is known that $Z_n = (Z(i,n); i = 1, \ldots, X_{n-1}$ and $X_{n-1} \in N^+)$ is a binary process. A special case of this binary process is the Bernoulli process with IID random variables. A special case of this counting process $X = (X_n, n \in N = (1, 2, \ \ldots, \ )$ is the binomial process with X_n distributed as binomial with parameters X_{n-1} and an identical success probability for all trials.

Let p_n be the kill probability of a defender assigned to an incoming missile at time n and d_n is the total number of defenders available at time n. The defense can choose any one of the three fire doctrines; [Eckler and Burr (1972)], at time n: (i) random, (ii) uniform, and (iii) general. A random fire results in the assignment of 0 to d_n defenders to an incoming missile, and a uniform fire results in the distribution of d_n defenders as uniformly as possible among incoming missiles. Fire doctrines other than random and uniform are called general.

For defenses against ballistic missiles, we show that the binary process exhibits correlation among the sequence of random variables and the counting process X is a supermartingale.

Neveu (1975) gives a detailed treatment of martingales and supermartingales. Throughout the paper, we say that X is a supermartingale with parameters $d = (d_1, d_2, \ldots, d_n, \ldots)$ and $p = (p_1, d_2, \ldots, d_n, \ldots)$.

2. The Stopping Time Problem and Bounds

The stopped time is dependent on X_0; we assume that $X_0 \ \leq \infty$. Let the stopping time be defined as follows: $T_0 = min(n : X_n \ \leq \alpha,\ \alpha \ \in N)$. Hence, T_0 is the time in which the destruction level is at most $X_0 - X_{T_0}$ and the penetration level is at most α. We will show that; under certain conditions, $P(T_0 \ < \ \infty) = 1$ and develop a dynamic recursive formula to get the value of T_0 that meets a certain defensive criterion. The optimal stopping time in this particular application of supermartingales is crucial in determining if the defense can maintain a certain proportion of destruction to incoming missiles for a given time. Also, the defense can identify the weakest time of engagement by redefining the optimal stopping time $\tau = min\{n :\mid X_{n-1} - X_n \mid > a,\ a \in N\}$.

We use the stochastic order relations; [Ross (1996), Chapter 9], between stochastic processes to show that the number of surviving ballistic missiles at time n for a supermartingale with parameters p and d using general fire doctrines is stochastically smaller than the number of surviving ballistic missiles at time n for a supermartingale with parameters p and d using random fire doctrines, and the former quantity is stochastically larger than the number of surviving ballistic missiles at time n for a supermartingale with parameters p and d using uniform fire doctrines. The same natural ordering applies for the optimal stopping times among these supermartingales. We also show that these supermartingales are stochastically increasing in X_0. Then, we use these stochastic ordering properties to derive bounds on the effectiveness of defenses against ballistic missiles.

REFERENCES

Eckler, A.R. and Burr, S.A. (1972). *Mathematical Models of Target Coverage and Missile Allocation. Military Operations Research Society, Washington, D.C.*

Neveu,J. (1975). *Discrete-Parameter Martingales. North-Holland, Amestrdam.*

Ross, S.M. (1996). *Stochastic Processes. Second Edition. Wiley, New York.*

RESUME

A martingales approach to analyze the effectiveness of defenses against ballistic missiles is presented. The problem of optimal stopping for these dynamic supermartingales is considered and a recursive formula is derived to find the optimal solution. Useful bounds on the effectiveness of those defenses are derived that are based on the notion of stochastic ordering. The results are illustrated by several examples.