A Comparison of Asymptotic Covariance Matrices of Three Estimators in Error Ridden Nonlinear Regression

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1 The model

We consider a model of the exponential family

$$f(y|\xi,\varphi) = \exp\{\varphi^{-1}(y\xi - C(\xi)) + c(y,\varphi)\},$$

where φ is an unknown scale parameter and $\xi = \xi(X,\beta)$ is a known function of the latent variable X and an unknown parameter vector β. Instead of X we observe

$$W = X + U,$$

where $U \sim N(0,\sigma_u^2)$ is the measurement error, the variance of which is supposed to be known. We assume the structural variant of the measurement error model, i.e., $X \sim N(\mu_x,\sigma_x^2)$, and X and U are independent.

2 Estimators

- The *naive* estimator of β, $\hat\beta_N$, is the solution to the likelihood score estimating equation of the model with X replaced by W :

$$\sum_{i=1}^{n} \psi(Y_i,W_i,\hat\beta_N) = 0 \quad , \quad \psi(Y,W,\beta) = [Y - C'\{\xi(W,\beta)\}]\frac{\partial\xi(W,\beta)}{\partial\beta} \ .$$

  $\hat\beta_N$ is asymptotically biased: $\hat\beta_N \to \beta_* \neq \beta$. Assuming ξ to be linear in β , $\hat\beta_N$ is asymptotically normal with asymptotic covariance matrix

$$\Sigma_N = \tfrac{1}{n}A_*^{-1}B_*A_*^{-1}$$
$$A_* = -E\left[\tfrac{\partial\psi^t(Y,W,\beta_*)}{\partial\beta_*}\right] \ , \ B_* = E\left[\psi(Y,W,\beta_*)\psi^t(Y,W,\beta_*)\right] \ .$$

- The *corrected quasi score* estimator of β, $\hat{\beta}_{CQS}$, is a solution to the corrected score estimating equation

$$\sum_{i=1}^{n} \psi_C(Y_i, W_i, \hat{\beta}_{CQS}) = 0 \ ,$$

where ψ_C is constructed such that

$$E[\psi_C(Y, W, \beta)|Y, X] = \psi(Y, X, \beta) \ .$$

$\hat{\beta}_{CQS}$ is strongly consistent and asymptotically normal with an asymptotic covariance matrix, Σ_{CQS}, that is similar to Σ_N except that ψ is replaced by ψ_C and β_* by β.

- The *structural quasi score estimator of β, $\hat{\beta}_{SQS}$, is the solution to the structural quasi score estimating equation, which is constructed in the following way. Let $m(W, \beta) = E(Y|W)$ and $v(W, \beta, \varphi) = V(Y|W)$. Then*

$$\begin{aligned} m(W, \beta) &= E[C'\{\xi(X, \beta)\}|W] \\ v(W, \beta, \varphi) &= V[C'\{\xi(X, \beta)\}|W] + \varphi E[C''\{\xi(X, \beta)\}|W] \ = \ A_1(W, \beta) + \varphi A_2(W, \beta) \ . \end{aligned}$$

These expressions can be evaluated, as $X|W \sim N(\mu(w), \tau^2)$ with $\mu(w) = W - \frac{\sigma_u^2}{\sigma_w^2}(W - \mu_w)$, $\tau^2 = \sigma_u^2 - \frac{\sigma_u^4}{\sigma_w^2}$. The estimating equations for β and φ are then given by

$$\begin{gathered} \sum_{i=1}^{n} \frac{\partial \hat{m}(W_i, \hat{\beta})}{\partial \beta} \hat{v}^{-1}(W_i, \hat{\beta}, \hat{\varphi})(Y_i - \hat{m}(W_i, \hat{\beta})) = 0 \\ \hat{\varphi} = \left\{ \sum_{i=1}^{n} \hat{A}_2(W_i, \hat{\beta}) \right\}^{-1} \sum_{i=1}^{n} \left[\{Y_i - \hat{m}(W_i, \hat{\beta})\}^2 - \hat{A}_1(W_i, \hat{\beta}) \right] \ , \end{gathered}$$

where the hat over m, v, A_j indicate that μ_w and σ_w^2 have been replaced by their estimates. $\hat{\beta}_{SQC}$ is strongly consistent and asymptotically normal with asymptotic covariance matrix

$$\begin{aligned} \Sigma_{SQS} &= \Phi^{-1} + \Phi^{-1}(\sigma_w^2 F_1 F_1^t + 2\sigma_w^{-4} F_2 F_2^t)\Phi^{-1} \\ \Phi &= E\left(v^{-1} \frac{\partial m}{\partial \beta} \frac{\partial m}{\partial \beta^t}\right) \ , \ F_j \ = \ E\left(v^{-1} \frac{\partial m}{\partial \beta} \frac{\partial m}{\partial \gamma_j}\right) \ , \ j = 1, 2 \ , \ \gamma_1 = \mu_w \ , \ \gamma_2 = \sigma_w^{-2} \ . \end{aligned}$$

3 Comparison of estimators

Despite the fact that the SQS method uses more information (namely the normality assumption for X) than CQS, which does not depend on any distributional assumptions about X, the following theorem holds for $\sigma_u^2 \to 0$.

Theorem: $\Sigma_{SQS} = \Sigma_{CQS} + O(\sigma_u^4)$.

We can also compare SQS or CQS to the naive method. These results are valid under rather general regularity assumptions. They hold true, in particular, for the polynomial and the Poisson regression models.

Reference: Kukush, A., Schneeweiss, H., Wolf, R.: *Comparing different estimators in a nonlinear measurement error model.* Discussion Paper 244, SFB 386, University of Munich.

Resumé

Nous comparons les matrices des covariances de trois estimateurs pour les paramètres d' un modèle de regression general avec des erreurs de mesure dans le variable indépendant.