# On numerical stability in characteristic function

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## 1. Abstract

Algorithm stability is used to support the notion that the CF is superior to the MGF in terms of numerically stable behavior. It is shown that there cannot exist computationally better tool than CF in terms of numerical stability. Uniqueness of the Vandermonde matrix with the perfect condition number is characterized for the numerical behavior of the CF.

**Key Words: moment generating function, characteristic function, condition number, Vandermonde matrix, qualitative computing, perturbation**

## 2. Ill-conditioning in MGF

Consider the continuous PGF function $p(t) = \mathrm{E}(t^X) = \sum_{j=0}^{n} p_j t^j$ by constructing a linear system:

$$\begin{pmatrix} 1 & t_0 & \dots & t_0^n \\ 1 & t_1 & \dots & t_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \dots & t_n^n \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} p(t_0) \\ p(t_1) \\ \vdots \\ p(t_n) \end{pmatrix} \tag{1}$$

with an $(n+1) \times (n+1)$ Vandermonde system matrix $V(t_0, \dots, t_n) = (t_{i-1}^{j-1})_{i,j}$ for each chosen manner of the nodes $\{t_0, \dots, t_n\}$. The system matrix $V = V(t_0, \dots, t_n)$ depends on only $(n+1)$ parameters $t_0, t_1, \dots, t_n$. Vandermonde matrices are related to an interpolation problem seeking a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ that interpolates the $n+1$ data $(t_0, p(t_0)), \dots, (t_n, p(t_n))$. The original motivation of the Vandermonde matrices came from unpleasant experiences with the computation of Gauss-type quadrature rules from the moments of the underlying weight function. (cf. Gautschi(1990)) Many statistical quadrature rules also require this Vandermonde system for finding its weight. Vandermonde matrices have the deserved reputation of being extremely ill-conditioned. The ill-conditioning is a consequence of the monomials being a poor basis for the polynomials on the real line. The condition number, showing how hard it is to be converted, is the reciprocal of the distance to the nearest singular problem with an infinite condition number. (cf. Higham(1996)) Ill-conditioning, measuring how much the linear system is sensitive to perturbations in the system matrix, results in inaccurate solution. Even if the algorithm is correct, rounding errors made when the system matrix is entered into the computer will perturb the solutions. Therefore, very high-precision arithmetic is required to obtain a high-order rule. (cf. Thisted(1988)) Obviously, the discretized MGF with system matrix $V(e^{t_0}, \ldots, e^{t_n})$ shall exhibit similar behavior as PGF. The cumulant generating function $K(t) = \log M(t)$ will also exhibit nothing but similar numerical behavior as the $M(t)$, based on the above discretization scheme.

## 3. A characterization of Vandermonde matrix with the perfect condition number

The situation changes if one allows complex nodes. First of all, the Fourier-transformed CF provides a corresponding linear system:

$$\begin{pmatrix} 1 & e^{\mathbf{i}t_0} & \ldots & e^{\mathbf{i}t_0 n} \\ 1 & e^{\mathbf{i}t_1} & \ldots & e^{\mathbf{i}t_1 n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\mathbf{i}t_n} & \ldots & e^{\mathbf{i}t_n n} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} \phi(t_0) \\ \phi(t_1) \\ \vdots \\ \phi(t_n) \end{pmatrix} \tag{2}$$

with an $(n+1) \times (n+1)$ *complex* Vandermonde system matrix $V(e^{\mathbf{i}t_0}, \ldots, e^{\mathbf{i}t_n}) = (e^{\mathbf{i}t_{i-1}}_{j-1})_{i,j}$ for $i, j = 1, \ldots, n+1$. If a particular harmonic Fourier frequencies $0 \leq t_j = 2\pi j/(n+1) < 2\pi$, for $j = 0, 1, \ldots, n$, are chosen, (cf. Luceño(1997)) then the normalized system matrix $V(e^{\mathbf{i}t_0}, \ldots, e^{\mathbf{i}t_n})/\sqrt{n+1}$ becomes unitary, and so it is norm-preserving isometry and its inverse $(V/\sqrt{n+1})^{-1}$ is, by definition, its conjugate transpose $\overline{V}^T/\sqrt{n+1}$. Therefore, the inverse of $V$ is $[V(e^{-\mathbf{i}t_0}, \ldots, e^{-\mathbf{i}t_n})]^{\mathrm{T}}/(n+1)$. Since the norm of any unitary matrix is equal to 1, so is the norm of its inverse. Hence the system matrix in Formula (**??**) achieves the (optimal) condition number 1 which *guarantees* backward stability, for safe matrix inversion.

Assume that an $(n+1) \times (n+1)$ Vandermonde matrix of the form

$$V(a_0, a_1, \ldots, a_n) = \begin{pmatrix} 1 & a_0 & \ldots & a_0^n \\ 1 & a_1 & \ldots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \ldots & a_n^n \end{pmatrix} \tag{3}$$

with complex numbers $a_0, a_1, \ldots, a_n$ has the perfect condition number, i.e., $\kappa(V) = \|V\|\|V^{-1}\| = 1$. From the well-known inequality: (cf. pp.124, Atkinson(1983))

$$\frac{\| \triangle x\|}{\|x\|} \leq \|V\|\|V^{-1}\| \frac{\|V(\triangle x)\|}{\|Vx\|} \tag{4}$$

for all $x$ and all perturbation $\triangle x$, we can get the equality

$$\|V(\triangle x)\| = \|V\|\| \triangle x\| \tag{5}$$

for all $\triangle x$. Applying the unit vector into the equality in Formula (??) we have the norm of $V$,

$$\|V\| = \sqrt{n+1}. \tag{6}$$

Hence, $V/\sqrt{n+1}$ is an isometry. Using the equalities in Formulae (??) and (??), we can calculate

$$\|\frac{V}{\sqrt{n+1}}(x)\| = \frac{1}{\sqrt{n+1}}\|V(x)\| = \frac{1}{\sqrt{n+1}}\|V\|\|x\| = \|x\|$$

From the well-known fact: A matrix is unitary if and only if it is isometric and surjective, (cf. pp.206, Kreyszig(1978)) $V/\sqrt{n+1}$ is unitary. Since $\overline{V}^T/\sqrt{n+1}$ is also unitary, we get

$$|a_j|^2 + |a_j|^4 + \cdots + |a_j|^{2n} = n$$

for $j = 0, 1, \ldots, n$. Thus, we know that $|a_j| = 1$ for all $j = 0, \cdots, n$. Since $V$ is nonsingular, $a_j$'s should be mutually distinct, using a well-known formula $\det(V) = \prod_{0 \le i,j \le n}(a_j - a_i)$. Let $a_j = e^{\mathbf{i}t_j}$ for $j = 0, 1, \cdots, n$. Using the equality

$$\frac{V}{\sqrt{n+1}}\Big(\frac{\overline{V}^T}{\sqrt{n+1}}\Big) = I,$$

we can show $a_0\bar{a}_j = e^{\mathbf{i}(t_0-t_j)}$, $a_1\bar{a}_j = e^{\mathbf{i}(t_1-t_j)}$, $a_2\bar{a}_j = e^{\mathbf{i}(t_2-t_j)}$, $\cdots$, $a_n\bar{a}_j = e^{\mathbf{i}(t_n-t_j)}$ are distinct roots of the equation $z^{n+1} - 1 = 0$. This completes the characterization. If we choose $t_0 = 0$, then $V$ has the form in Formula (??) where $e^{\mathbf{i}t_j}$'s are distinct $n$th roots of unity. In this case, this type of the matrix $V$ is a Fourier matrix. (cf. Higham(1996))

## 4. Concluding Remarks

The system matrices generated by the MGF and the CF have similar form of the Vandermonde matrix. If *any* Vandermonde matrix in Foumula (??) has the perfect condition number, then it has a form of CF system matrix in Formula (??) whose main elements are points on the unit circle in the complex plane on which the points are equispaced. Furthermore, if the points are rotated clockwise by one of them, the resulting points are still distinct complex roots of unity. It is shown that there cannot exist numerically better tool than the CF, since any condition number of a matrix cannot be below 1. Applications, such as polynomial interpolation and statistical quadrature, can be based on the complex roots of unity.

Some other statistical tools than the MGF in this article, already setup in a mathematical sense, might have serious instability in finite-digit implementation. Most modern computer architecture cannot avoid this phenomenon. Classical mathematical statistics may not protect one from finite precision computation.

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