Time series chain graphs and Granger causality

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1. Time series chain graphs

Over the last years there has been growing interest in graphical models as a general framework to describe and infer causal relations (e.g. Pearl 2000). Here, we discuss the properties of time series chain graphs, in which each variable $X_a(t)$ at a specific time t is represented by a separate vertex in the graph, and their relation to the concept of Granger-causality (Granger, 1969).

Definition (Time series chain graph) The time series chain graph (TSC-graph) of the stationary process $\{X(t)\}$ is the chain graph $G_{\mathrm{TS}} = (V_{\mathrm{TS}}, E_{\mathrm{TS}})$ with $V_{\mathrm{TS}} = \{(a,t) | a \in V, t \in \mathbb{Z}\}$ and edge set E_{TS} such that
(i) $(a, t-u) \longrightarrow (b,t) \notin E_{\mathrm{TS}} \iff u \leq 0$ or $X_a(t-u) \perp X_b(t) \,|\, \bar{X}_V(t) \backslash \{X_a(t-u)\}$,
(ii) $(a, t-u) — (b,t) \notin E_{\mathrm{TS}} \iff u \neq 0$ or $X_a(t) \perp X_b(t) \,|\, \bar{X}_V(t) \cup \{X_{V\backslash\{a,b\}}(t)\}$.

Eichler (2001) introduced Granger-causality graphs for multivariate time series in which each component of the time series is represented by only one vertex and the arrows connecting the vertices reflect Granger-causality. There is a simple relation between the two graphs:

Proposition (Aggregation) *Let G_{C} and G_{TS} be the causality graph and the TSC-graph, respectively, of the process $\{X(t)\}$. Then we have*
(i) $a \longrightarrow b \notin E_{\mathrm{C}} \iff (a, t-u) \longrightarrow (b,t) \notin E_{\mathrm{TS}} \;\; \forall u > 0 \; \forall t \in \mathbb{Z},$
(ii) $a — b \notin E_{\mathrm{C}} \iff (a,t) — (b,t) \notin E_{\mathrm{TS}} \; \forall t \in \mathbb{Z}.$

These time series graphs visualize the pairwise interaction structure between the components of the process. In other words, they reflect the pairwise Markov properties of the process. The interpretation of the graphs is enhanced by global Markov properties which relate the separation properties of a graph to conditional orthogonality or causality relations between the components of the process (Eichler, 2001).

2. Inference for time series chain graphs

We first discuss the case of fitting VAR(p) models restricted with respect to a time series chain graph. In practice the true TSC-graph is unknown and one has to apply model selection strategies to find an optimal TSC-graph. More precisely, let $G = (V_{\mathrm{TS}}, E)$ be a graph from the class $\mathcal{G}_{\mathrm{TS}}(p)$ of all time series chain graphs whose edges have at most lag p and which are invariant under translation in the sense that for all $a, b \in V$, and all $t, s, u \in \mathbb{Z}$ we have $(a, t-u) \longrightarrow (b,t) \notin E$ if and only if $(a, s-u) \longrightarrow (b,s) \notin E$ and $(a,t) — (b,t) \notin E$ if and only if $(a,s) — (b,s) \notin E$.

We consider Gaussian VAR(p) models of the form

$$X(t) = A(1)X(t-1) + \ldots + A(p)X(t-p) + \varepsilon(t), \qquad \varepsilon(t) \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma_G)$$

where the parameters $A_G = (A(1), \ldots, A(p),)$ and $K_G = \Sigma_G^{-1}$ satisfy the constraints $A_{ba}(u) = 0$ if $(a,t) \longrightarrow (b, t+u) \notin E$ and $K_{ab} = 0$ if $(a,t) — (b,t) = 0$. We call a vector autoregressive model with these constraints on the parameters a VAR(p, G) model.

Differentiating the conditional log-likelihood function of the Gaussian VAR(p, G) model we obtain equations for the maximum likelihood estimate which can be solved numerically by an iterative algorithm (Dahlhaus and Eichler, 2001). Furthermore we can use model selection criteria for selection of the optimal VAR(p, G) model among all models with $1 \leq p \leq P$ and $G \in \mathcal{G}_{\text{TS}}(p)$. Since in practice the number of possible models is too large one has to use special model selection strategies.

Alternatively we can estimate the partial correlations

$$\pi_{ba}(u) = \text{corr}\,\big(X_b(t), X_a(t-u)\big|\bar{X}_V(t)\backslash\{X_a(t-u)\}\big),$$
$$\pi^{\circ}_{ba} = \text{corr}\,\big(X_b(t), X_a(t)\big|\bar{X}_V(t) \cup \{X_{V\backslash\{a,b\}}(t)\}\big),$$

where the partial correlations are taken to be the correlations about the linear projection. The time series chain graph G_{TS} $\{X(t)\}$ can be characterized in terms of these partial correlations,

$$(a, t-u) \longrightarrow (b,t) \notin E_{\text{TS}} \iff \pi_{ba}(u) \neq 0,$$
$$(a,t) — (b,t) \notin E_{\text{TS}} \iff \pi^{\circ}_{ba} \neq 0.$$

In Dahlhaus and Eichler (2001) an iterative scheme for the estimation of these correations has been suggested. The estimators can then be used for testing for the existence of an edge in the TSC-graph G_{TS}.

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