Join trees for combining decomposable graphical models

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Fienberg and Kim (1999) consider a problem of combining conditional graphical log-linear structures and derive a combining rule of them based on the relation between the log-linear model and its conditional version. In the graph combination, the conditioning variable is added to the set, say A, of the variables that are involved in the conditional structures and so each graph-combination ends up with a new graph that involves the conditioning variable in addition to the set A.

While Fienberg and Kim (1999) consider graphs of conditional graphical log-linear models, we will consider the problem of combining graphs of marginal (as against conditional) graphical models of various types of random variables under the condition that the graph of the model of the variables that are involved in at least one of the marginal models is decomposable. Some attractive features of the decomposable graph are that it is triangulated (Darroch, Lauritzen, Speed (1980); Leimer (1989)), that decomposability is preserved in graph-collapsing and that every directed acyclic graph is convertible into a decomposable graph (Lauritzen and Spiegelhalter, 1988). In this regard among others, we will confine ourselves on decomposable graphs in this article.

Suppose that we are given a pair of simple marginals, [12][23] and [24][25], where only one variable is shared. In this case, we have a longer list of possible joint model structures as follows:

$$\begin{array}{l}[12][24][23][25], [124][23][25], [124][23][35], [124][25][35], [124][235],\\ [125][23][34], [125][24][34], [125][234].\end{array} \tag{1}$$

Model structures [124][235] and [125][234] are maximal in the sense of set inclusion among these eight models.

It is important to note that some variable(s) are independent of the others conditional on X_2 in each of the two pairs and in all the models in (**??**). That conditional independence takes place conditional on the same variable in the marginal structures and also in the joint structures plays a key role in combining graphical models.

We address the issue of combining graphical model structures and so we can not help using independence graphs and related theories to derive desired results with more clarity and refinement. Toward that end, we introduce a new graphical concept called a graph connector or connector for short, which are defined as connectors of cliques. A minimal connector in a graph has a very attractive property that a minimal connector in a subgraph of a graph $\mathcal{G}$ is also a minimal connector in $\mathcal{G}$.

In combining a pair of graphical models, we make use of the property and combine them by grafting the minimal connectors of a graph to certain points of the other graph.

When there are many minimal connectors involved, we will consider a join tree of only minimal connectors, one for each graphical model to be combined. When the model contains a number of variables and its structure is complicated, such a join tree will be a useful tool for model combination, because the minimal connectors are to be preserved in the combined model and it is easier to find grafting locations from the structure of the join trees.

The method proposed for model combination is applied to artificial examples for illustration and it is shown that the join tree approach saves much of our time and effort in finding grafting locations and thus the whole process of model combination.

References

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