The Pricing of Catastrophe Reinsurance Contract Using the Cox Process and an Equivalent Martingale Probability Measure

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1. Introduction

Insurance companies have used reinsurance contracts to hedge themselves against losses from catastrophic events. Most commonly used reinsurance contract is excess of loss contract. The excess of loss reinsurance premium at time 0, assuming that interest rates to be constant, is

$$E\left[\sum_{i=1}^{N_t}\left\{Max\left(Z_i-b,\ 0\right)\right\}\right]=E\left(N_t\right)\cdot E\left[Max\left(Z_i-b,\ 0\right)\right]=E\left(N_t\right)\cdot E\left[\left(Z_i-b\right)^+\right] \tag{1.1}$$

where N_t is the number of claims up to time t, Z_i is the claim amount, which are assumed to be independent and identically distributed with distribution function $H(z)$ $(z>0)$ and b is a retention limit.

During the last decade, the high level of worldwide catastrophe losses in terms of frequency and severity had a marked effect on the reinsurance market. The catastrophes such as Storm Daria (Europe 1990), Hurricane Andrew (USA 1992) and the Kobe earthquake (Japan 1995) have impacted the profitability and capital bases of reinsurance companies. Some of these companies have withdrawn from the market while others have reduced the level of catastrophe cover they are willing to provide (see Booth (1997)). Therefore the reinsurance market emphasises the need for an appropriate pricing model for reinsurance contracts. In other word, the above expectation (1.1) should be calculated under an appropriate probability measure.

2. The Cox process and shot noise process

For catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a doubly stochastic Poisson process, or the Cox process (see Cox (1955), Bartlett (1963), Serfozo (1972), Grandell (1976, 1991), Bremaud (1981) and Lando (1994)),

$$\Pr\left\{N_{t_2}-N_{t_1}=k \mid l_s; t_1\le s\le t_2\right\}=\frac{\exp\left(-\int_{t_1}^{t_2} l_s\,ds\right)\left(\int_{t_1}^{t_2} l_s\,ds\right)^k}{k!}. \tag{2.1}$$

Claims arising from catastrophic events depend on the intensity of natural disasters (e.g. flood, windstorm, hail, and earthquake). One of the processes that can be used to measure the impact of catastrophic events is the shot noise process (see Cox & Isham (1980,1986) and Klüppelberg & Mikosch (1995)). Therefore the shot noise process can be used as the parameter of the doubly stochastic Poisson process to measure the number of claims due to catastrophic. We will adopt the shot noise process used by Cox & Isham (1980):

$$l_t = l_0 e^{-dt} + \sum_{\substack{\text{all } i\\ s_i\le t}} y_i e^{-d(t-s_i)} \tag{2.2}$$

where i is a catastrophe, λ_0 is initial value of λ, y_i is jump size of catastrophe i (i.e. magnitude of contribution of catastrophe i to intensity) with distribution function $G(y)$ $(y>0)$ where $E(y)<\infty$, s_i is time at which catastrophe i occurs where $s_i<t<\infty$, δ is exponential decay which never reaches zero and ρ is the rate of catastrophe jump arrival.

The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. Therefore, assuming that three parameters of the shot noise process are time dependent, the generator of the process (X_t, N_t, λ_t, t) acting on a function $f(x,n,\lambda,t)$ belonging to its domain is given by

$$\begin{aligned} \mathcal{A} f(x,n,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \lambda[f(n+1,\lambda,t) - f(n,\lambda,t)] - \delta(t)\,\lambda\frac{\partial f}{\partial \lambda} \\ &\quad + \rho(t)\Big[\int_0^\infty f(x,n,\lambda+y,t)\,dG(y;t) - f(x,n,\lambda,t)\Big] \end{aligned} \tag{2.3}$$

where $X_t = \int_0^t \lambda_s ds$.

3. No-arbitrage, the Esscher transform and change of probability measure

Sondermann (1991) introduced the non-arbitrage approach for the pricing of reinsurance contracts. He proved that if there is no arbitrage opportunities in the market, reinsurance premiums are calculated by the expectation of their value at maturity with respect to a new probability measure and not with respect to the original probability measure. This new probability measure is called the equivalent martingale probability measure. Therefore the existence of an equivalent martingale probability measure is equivalent to the assumption of no arbitrage opportunities in the market.

We will examine an equivalent martingale probability measure obtained via the Esscher transform (see Gerber & Shiu (1996)). As a result of changing measure, the generator $\mathcal{A}^*$ of the process (X_t, N_t, λ_t, t) acting on a function $f(x,n,\lambda,t)$ with respect to the equivalent martingale probability measure, assuming that $\delta(t)=\delta$, changes to

$$\begin{aligned} \mathcal{A}^* f(x,n,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \theta^*\lambda\Big\{\int_0^\infty f(x,n+1,\lambda,t) - f(x,n,\lambda,t)\Big\} - \delta\lambda\frac{\partial f}{\partial \lambda} \\ &\quad + \rho^*(t)\Big\{\int_0^\infty f(x,n,\lambda+y,t)\,dG^*(y;t) - f(x,n,\lambda,t)\Big\} \end{aligned} \tag{3.1}$$

where $\rho^*(t)=\rho\,\hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t)=\dfrac{\exp(-\gamma^* e^{\delta t}y)\,dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. In other words, the risk-neutral Esscher measure is the measure with respect to which N_t becomes the Cox process with parameter $\theta^*\lambda_t$ where three parameters of the shot noise process λ_t are δ, $\rho^*(t)=\rho\,\hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t)=\dfrac{\exp(-\gamma^* e^{\delta t}y)\,dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. Therefore the probability generating function of the distribution of N_t with respect to the equivalent martingale probability measure, assuming that the shot noise process is asymptotic (stationary) and that the jump size follows an exponential distribution, i.e. $g(y)=\alpha e^{-\alpha y}$, $y>0$, $\alpha>0$, becomes

$$E^*(q^{N_{t_2}-N_{t_1}}) = \left(\frac{g^* e^{dt_1} + a e^{-d(t_2-t_1)}}{g^* e^{dt_1} + a + \frac{q^*(1-q)}{d}(1-e^{-d(t_2-t_1)})} \right)^{\frac{r}{d}} \left(\frac{g^* e^{dt_1} + a + \frac{q^*(1-q)}{d}(1-e^{-d(t_2-t_1)})}{g^* e^{dt_1} + a e^{-d(t_2-t_1)}} \right)^{\frac{ar}{da+q^*(1-q)}} \tag{3.2}$$

where $0 \le q \le 1$, $q^* \ge 1$ and $-ae^{-dt} < g^* \le 0$.

In practice, the reinsurer will calculate the premium of an excess of loss contract using $q^* > 1$ and $g^* < 0$. This results in the reinsurer assuming that there will be a higher value of claim intensity itself, a higher value of the damage caused by the catastrophe and more catastrophes occurring in a given period of time. These assumptions are necessary, as the reinsurer wants compensation for the risks involved in operating in incomplete market. The reinsurer also aims to maximise their shareholders' wealth by earning profits rather than operating at breakeven point where premiums are equal to expected claims that is calculated with respect to the original probability measure. If $q^* = 1$ and $g^* = 0$, then net premium is calculated which should cover the expected losses over the period of contract. Therefore we can consider q^* and g^* as security loading factors by which gross premium that should be finally charged, will be calculated. The insurance companies' attitude towards risk determines how to levy the security loading on the net premium (i.e. which equivalent martingale probability measure should be used to obtain the gross premium using the combination of q^* and g^*).

4. Pricing of an excess of loss reinsurance

The excess of loss reinsurance gross premium at time 0 with respect to an equivalent martingale probability measure, P^*, with gamma claim size distribution i.e. $h(z) = \frac{b^j z^{j-1} e^{-bz}}{(j-1)!}$, $z > 0$, $b > 0$, $j \ge 1$, is

$$E^*(N_t) \cdot E\left[(Z_i - b)^+ \right] = \left\{ \frac{q^* r}{da} t - \frac{q^* r}{d^2 a} \cdot \ln\left(\frac{g^* e^{dt} + a}{g^* + a} \right) \right\} \left\{ \frac{j}{b} \int_b^\infty \frac{b^{j+1} z^j e^{-bz}}{j!} dz - b \int_b^\infty \frac{b^j z^{j-1} e^{-bz}}{(j-1)!} dz \right\}. \tag{4.1}$$

The parameter values used to expand (3.2), using the *MAPLE* algebraic manipulation, with respect to q are $q^* = 1.1$, $g^* = -0.1$, $a = 1$, $d = 0.3$, $r = 4$, $t = 1$. The parameter values used to calculate (4.1) are n : 1 ~ 41, $j = 1$, $b = 1$, $b = 0$, 0.025, 0.5, 0.75, 1, 1.25, 1.5 and $E(Z) = 1$. By computing (4.1), using *S-Plus* the calculation of the excess of loss reinsurance gross premiums for catastrophic events at each retention level b are shown in Table 4.1.

Table 4.1 Excess of loss reinsurance gross premium

Retention level b	Reinsurance gross premium
0.00	16.61
0.50	14.81
1.00	6.11
1.50	3.71
2.00	2.25
2.50	1.36
3.00	0.83

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RESUME

Ji-Wook Jang obtained his B.A. in Business Administration from Sogang University, Seoul in 1991. After studying Actuarial Science for Master of Science at the City University, London, he completed his Ph.D. at the London School of Economics and Political Science ('LSE') in 1998 in Statistics. Before he started his doctorate with LSE, he went to work for LG Securities International Limited, London in 1994 as an assistant to head trader and the City University, London in 1993 as a researcher for a paper prepared for Scottish Amicable, Glasgow.

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