On Improving Quality with Multiple Responses

Chih-Hua Chiao
Soochow University, Department of Business Mathematics
56, Kuei-Yang Street, Section 1,
Taipei, Taiwan
chchiao@bmath.scu.edu.tw

Michael Hamada
Los Alamos National Laboratory, Statistical Sciences
Los Alamos NM 87545, USA

1. Abstract

Statistically designed experiments have been employed extensively to improve product or process quality and to make it robust. In this paper, we consider such experiments with correlated multiple responses whose means, variances and correlations depend on the experimental factors. Analysis of these experiments consists of modeling these distributional parameters in terms of the experimental factors and finding factor settings which maximize the probability of being in a specification region, i.e., all responses are simultaneously meeting their respective specifications. The proposed procedure is illustrated with three experiments from the literature.

2. Introduction

In this paper, a simple method is proposed to properly handle multiple responses. First, the parameters of the multiple response distribution are modeled in terms of the experimental factors. Then, factor settings that optimize a suitable criterion are found. The criterion considered in this paper is the probability that all responses simultaneously meet their respective specifications, a criterion that is easily interpretable.

For m responses, let $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_m)^T$ denote the $m \times 1$ vector of multiple responses or an m dimensional multivariate response. Each component response Y_i has a specification (l_i, u_i) consisting of lower and upper limits within which it is desirable for Y_i to be. Using the component specifications, we define a *specification region* S for the multivariate response to be the m dimensional rectangle whose sides are the m component specifications. Given a specification region, a measure of quality is the probability that the m component responses are simultaneously meeting their respective specifications or the proportion of conformance

$$P(\mathbf{Y} \in S). \tag{1}$$

We assume that $\mathbf{Y}$ follows an m dimensional multivariate normal distribution with mean $\boldsymbol{\mu}= (\mu_1, \mu_2, \cdots, \mu_m)^T$ and variance-covariance matrix $\boldsymbol{\Sigma}$ whose diagonal elements are the variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_m^2$ and off-diagonal elements are the covariances $\rho_{k\ell}\sigma_k\sigma_\ell$, $1 \leq k < l \leq m$, where $\rho_{k\ell}$ is the correlation between Y_k and Y_l. Assuming the multivariate normal distribution for $\mathbf{Y}$ holds, the proportion of conformance (1) can be evaluated for a given $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. In order to improve quality or make quality robust, one needs to first identify what variables impact these distributional parameters and then find the values of these variables which achieve these goals.

3. Distributional Parameter Models

Let $\mathbf{x}$ denote a row vector of covariates associated with a set of factors which may include an intercept, main effects, two-factor interactions, etc. Consider the following models for the distribution parameters of the m dimensional multivariate response:

$$\mu_k = \mathbf{x}\boldsymbol{\alpha}_k, \quad k = 1, \cdots, m, \tag{2}$$

$$\log(\sigma_k^2) = \mathbf{x}\boldsymbol{\beta}_k, \quad k = 1, \cdots, m, \tag{3}$$

and

$$\tanh^{-1}(\rho_{k\ell}) = \mathbf{x}\boldsymbol{\gamma}_{k\ell}, \quad 1 \leq k < l \leq m. \tag{4}$$

Note that $\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_{k\ell}$ are column vectors. The nonzero components of $\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_{k\ell}$ determine the impact of the associated variables.

4. Analysis Methodology

The form of the covariate vector $\mathbf{x}$ depends on the type of experiment. For fractional factorial designs, the $\mathbf{x}$ consists of covariates corresponding to an intercept, main effects and two-factor interactions involving the factors $F_1, \ldots, F_r$. For quality improvement, the experimental goal is to maximize the proportion of conformance (1), the proportion of products which simultaneously meet all specifications. For robust quality improvement, the experimental goal is to maximize the proportion of conformance (1) while accounting for variation in the noise factors. To do this, one must first identify the distributional parameter models (2) $-$ (4) for the multivariate normal response and use them to find the factor settings that accomplish the goal. Thus the proportion of conformance depends on the factor levels $\mathbf{x}$ which is expressed as

$$P(\mathbf{Y} \in S|\mathbf{x}). \tag{5}$$

Consider experiments in which there are n replicates at each experimental run, i.e., each factor setting in the experimental design. We propose a simple method that first estimates the distributional parameters of the multivariate normal response, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, at each run and then identifies the experimental factor models (2)-(4). The experimental factor models (2)-(4) can then be fit using the $(\hat{\mu}_k, \hat{\sigma}_k^2, \hat{\rho}_{k\ell})$ to obtain least square estimates (LSE's) for the factorial effects $\mathbf{\Theta}$ given in these models.

Based on the estimates of the significant effects, factor settings can then be found that maximize the proportion of conformance (5) in quality improvement. As in any data analysis its worth checking model assumptions. Andrews et al. (1973) provide a comprehensive review of tests for multivariate normality.

Example: Wheel Cover Component Experiment

Harper, Kosbe and Peyton (1987) reported on an experiment to find the optimum combination of injection molding parameters to minimize the imbalance of a plastic wheel cover component. Seven factors thought to be potentially important to the component's balance are listed in Table 1 as well as the levels for each factor denoted by -1 and $+1$. The relevant quality characteristics of the wheel cover component are the total weight (Y_1 in grams) and the balance (Y_2 in inch-ounces) of the component. The two-sided specification region for Y_1 and Y_2 is defined by (710, 715) and (0.3, 0.4), respectively. The data for eight runs of a 2^{7-4} fractional factorial

Table 1. Factors and Levels for Wheel Cover Component Experiment

	Level	
Factor	-1	$+1$
F_1: mold temperature	80	110
F_2: close time	16	21
F_3: booster time	1.88	1.7
F_4: plunger time	4	8
F_5: pack pressure	1300	1425
F_6: hold pressure	1100	700
F_7: barrel	490	505

design are shown in Table 2 in which five components were measured in each run. Only main effects of the seven factors can be studied in this experiment. Estimates for $\mathbf{\Theta}$ which includes all main effects $\{\alpha_p, \beta_p, \gamma_p\}$ for $p = 1, \cdots, 7$ and their intercepts, respectively, were obtained by least squares. Using these standard deviations for the LSE's of the effects for $\hat{\sigma}_k$ (k=1,2) and $\tanh^{-1}\hat{\rho}$ and Lenth's approach for $\hat{\mu}_k$ (k=1,2), we identified the following experimental factor models:

$$
\begin{aligned}
\hat{\mu}_1 &= 720.763 + 1.873x_1 + 5.318x_5 - 3.408x_7, \\
\hat{\mu}_2 &= 0.967 + 0.113x_1 + 0.328x_5 - 0.174x_7, \\
\log \hat{\sigma}_1^2 &= 0.944 - 0.509x_2 + 1.189x_4 + 1.196x_5 - 0.487x_7, \\
\log \hat{\sigma}_2^2 &= -4.797 - 0.692x_2, \\
\tanh^{-1}(\hat{\rho}_{12}) &= -0.116 + 0.490x_1 - 0.505x_5,
\end{aligned} \tag{6}
$$

Table 2. Experimental Design and Bivariate Responses for Wheel Cover Component Experiment

Run	F_1	F_2	F_3	F_4	F_5	F_6	F_7	**Y**	Replicate				
1	−1	−1	−1	−1	−1	−1	−1	Y_1	711.9	713.4	712.3	712.4	711.9
								Y_2	.59	.59	.47	.71	.63
2	−1	−1	−1	+1	+1	+1	+1	Y_1	725.0	720.1	711.8	723.9	720.9
								Y_2	.70	.91	1.13	.79	.78
3	−1	+1	+1	−1	−1	+1	+1	Y_1	711.6	711.7	711.3	712.1	711.7
								Y_2	.56	.44	.46	.53	.46
4	−1	+1	+1	+1	+1	−1	−1	Y_1	733.7	724.1	732.0	732.7	733.3
								Y_2	1.50	1.55	1.38	1.45	1.45
5	+1	−1	+1	−1	+1	−1	+1	Y_1	725.4	721.6	722.6	723.1	721.1
								Y_2	1.25	1.36	1.51	1.22	1.25
6	+1	−1	+1	+1	−1	+1	−1	Y_1	728.7	721.1	722.9	723.0	719.7
								Y_2	1.17	.97	.98	.97	.73
7	+1	+1	−1	−1	+1	+1	−1	Y_1	726.6	731.4	731.4	729.6	731.3
								Y_2	1.52	1.58	1.61	1.40	1.57
8	+1	+1	−1	+1	−1	−1	+1	Y_1	714.3	714.4	713.6	716.3	714.6
								Y_2	.57	.51	.44	.44	.56

where x_1, x_2, x_4, x_5 and x_7 are the covariates corresponding the five significant factors. Using the experimental factor models, the proportion of conformance (5) can be evaluated for all 128 ($= 2^7$) possible factor settings with each factor taking on levels -1 and $+1$. Based on values the proportion of conformance, the optimal setting $\mathbf{x}_{\text{opt}} = (-1, +1, -, -1, -1, -, +1)$, where an insignificant factor is denoted by $-$ and whose $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\rho}) = (710.16, 0.35, 0.30, 0.06, -0.10)$. The proportion of conformance at the optimal setting is 0.515.

REFERENCES

Andrews, D.F., Gnanadesikan, R., and Warner, J.L. (1973). Methods for Assessing Multivariate Normality in *Multivariate Analysis* (Vol. 3), Ed. P.R. Krishnaiah, 825-840.

Harper, D., Kosbe, M., and Peyton, L. (1987). Optimization of Ford Taurus Wheel Cover Balance (by Design of Experiments - Taguchi's Method) in *Fifth Symposium on Taguchi Methods*, 527-539.

RESUME

Chih-Hua Chiao is an Associate Professor in the Department of Business Mathematics. Michael Hamada is a Technical Staff Member in Statistical Sciences, Los Alamos National Laboratory.