Comparison of Generalized Markov Branching Processes

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The (one-dimensional) Generalized Markov Branching Processes (GMBP) is a Z_+-valued continuous time Markov chain whose infinitesimal q-matrix $Q=\{q_{ij}\}$ is given by

$$q_{ij} = \begin{cases} w_i b_{j-i+1} & \text{if} \quad j \geq i-1, i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $b_j \geq 0$ ($j \neq 1$), $-b_1 = \sum_{j\neq 1} b_j < +\infty$ $\quad w_0=w_1=1$, $w_i>0$ ($i\geq 2$).

Although there exist substantial interest in this kind of dependent branching processes, few properties have been known except the well-known ordinary Markov branching processes (MBP) and the $M^X/M/1$ queue (killed at idle time) where $w_n=n$ ($n\geq 1$) and $w_n\equiv 1$, respectively. Now, denote $Q=WB$ where W and B should be self-explained. Our basic idea is to compare two processes corresponding to $Q^{(1)}=W^{(1)}B$ and $Q^{(2)}=W^{(2)}B$. The following result is then obtained.

Theorem 1. Suppose $W^{(1)}\leq W^{(2)}$, then the corresponding Markov transition functions (not necessarily honest) $P^{(1)}(t)=\{p_{ij}^{(1)}\}$ and $P^{(2)}(t)=\{p_{ij}^{(2)}(t)\}$ satisfy $p_{i0}^{(1)}(t) \leq p_{i0}^{(2)}$ $(\forall i \geq 0, t \geq 0)$ and $\sum_{j=0}^{\infty} p_{ij}^{(1)}(t) \geq \sum_{j=0}^{\infty} p_{ij}^{(2)}(t)$ $(i \geq 0, t \geq 0)$.

By comparing the less-known GMBP with the well-investigated MBP and $M^X/M/1$ queue, we are able to obtain many properties of GMBP. In particular, the following results can be obtained for a very large class of GMBP.

Theorem 2. Under some mild conditions, we have

(i) If $\sum_{j=2}^{\infty}(j-1)b_j \leq b_0$ the extinction probability of GMBP is 1 while if $\sum_{j=2}^{\infty}(j-1)b_j > b_0$, the extinction probability, starting from i, is q^i where $q<1$ is the smallest root of B(s)=0 and B(s) is the generating function of $\{b_j; j\geq 0\}$.

(ii) If $\sum_{j=2}^{\infty}(j-1)b_j > b_0$ and the GMBP is dishonest, then

$$p_{i0}(t) \uparrow q^i \quad (i \geq 1) \text{ and } \sum_{j=0}^{\infty} p_{ij}(t) \downarrow q^i \quad (i \geq 1)$$

It is interesting to note that the conclusions in Theorem 2 are independent of the structure W. Also, the probability meaning of these conclusions are quite clear.